Hacettepe University

Mathematics I
Exam Solutions Manual

1. Introduction

Dear math learners, this document comprises the midterm and final exams of the course *Mathematics I* with the code *MAT123*. A sample of each preserved exam held in the past has been gathered from students in the Department of Electrical and Electronics Engineering. The exams are categorized into two main exams: Midterms and Finals. Every midterm exam is identified as a *Midterm* exam, whereas the *Final*, *Resit*, and *Makeup* exams are considered to be final exams throughout the document. The solutions are checked through various platforms, including *Desmos*, *GeoGebra*, and *WolframAlpha*.

Should you find a mistake with the solution or need consultation, please get in touch with the author via the following:

Author's name: Baturay Kafkas
Author's email: baturaykafkas@hacettepe.edu.tr
Author's personal email: kfksbtry@gmail.com

2. Motivation

- To help learners understand the fundamentals of calculus,
- To make students have the knowledge of mathematical symbols and notations,
- To help students prepare for exams with proper writing skills,
- To make students gain ideas for advanced math courses,
- To help students have expertise in academic writing.

3. Exams and Solutions

There are a total of 13 midterm exams and 12 final exams.

Exam	Date	Content Page	Solution Page
Midterm	06/04/2012	4	31
Midterm	08/05/2012	5	35
Midterm	15/11/2012	6	39
Midterm	19/12/2012	7	43
Midterm	10/12/2015	8	46
Midterm	05/12/2016	9	50
Midterm	24/07/2018	10	54
Midterm	04/11/2019	11	59
Midterm	30/11/2020	12	66
Midterm	08/12/2021	13	70
Midterm	23/11/2022	14	76
Midterm	01/11/2023	15	80
Midterm	02/12/2024	16	84

Exam	Date	Content Page	Solution Page
Final	29/05/2012	18	90
Final	23/01/2013	19	94
Makeup	28/01/2015	20	99
Final	23/08/2015	21	104
Final	12/01/2016	22	112
Final	13/08/2018	23	120
Final	18/01/2021	24	127
Final	07/01/2022	25	131
Final	11/01/2023	26	135
Final	13/01/2023	27	139
Final	09/01/2025	28	143
Makeup	31/01/2025	29	147

MIDTERMS

2011-2012 Spring
MAT123-[Instructor] Midterm I
(06/04/2012)
Time: 15:00 - 16:45
Duration: 105 minutes

1. Find the equation of the tangent line to the curve $x^2 + 2xy + y^2 = 4$ at the point $(3, -1)$.

2. Find the point on the parabola $y = \sqrt{x}$ which is the closest to the point $(2, 0)$.

3. Evaluate the limit, if it exists, and explain your answer. Do not use L'Hôpital's rule.

(a) $\lim\limits_{x \to 0} \sqrt{x^2 + 2x^3} \sin\left(\frac{1}{x}\right)$ (b) $\lim\limits_{x \to 3} \frac{x^2 - 9}{x^2 - x - 6}$ (c) $\lim\limits_{x \to -\infty} \sqrt{x^2 - 4x} + x$

(d) $\lim\limits_{h \to 0} \frac{(1 + h)^{123} - 1}{h}$

4. Find the derivatives of the following functions.

(a) $f(x) = x^{\cos(x^3)}$ (b) $f(x) = \tan\left(e^{2x} \sin(3x)\right)$

(c) Find the second derivative $f''(x)$ of $f(x) = \ln\left(\frac{x^2}{x^2 + 4}\right)$.

5. For the function $f(x) = \frac{1}{x^2 - 4}$,

(a) Find the vertical and horizontal asymptotes.

(b) Find the intervals of increase or decrease.

(c) Find the local maximum and minimum values, if any.

(d) Find the intervals of concavity and the inflection points, if any.

(e) Sketch the graph of f.

2011-2012 Spring
MAT123-[Instructor] Midterm II
(08/05/2012)
Time: 15:00 - 16:45
Duration: 105 minutes

1. Consider the region bounded by the curve $y = x^2$, the x-axis, and the line $x = 2$, where $x \geq 0$.

(a) Find the volume of the solid generated by revolving the region about the x-axis by the disk method and sketch the solid.

(b) Find the volume of the solid generated by revolving the region about the y-axis by the shell method and sketch the solid.

2. Evaluate the following integrals.

(a) $\displaystyle\int_0^3 \left|x^2 - 1\right| dx$ (b) $\displaystyle\int \frac{1}{x^2 + 2x + 1} dx$ (c) $\displaystyle\int \frac{1}{x^2 + 2x + 2} dx$

(d) $\displaystyle\int \frac{1}{x^2 + 3x + 2} dx$ (e) $\displaystyle\int_{-\pi/6}^{0} \sqrt{1 - \cos(6x)}\, dx$

3. Determine whether the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1 + x^2} dx$ is convergent or divergent. Evaluate if the integral is convergent.

4. Evaluate the following limits. Explain all your work and write clearly.

(a) $\displaystyle\lim_{x \to 0} \frac{3^x - 1}{5^x - 1}$ (b) $\displaystyle\lim_{x \to 0} \frac{\int_0^{x^2} \sin t\, dt}{x}$

5. Determine whether each sequence converges or diverges. Evaluate the limit of each convergent sequence. Explain all your work and write clearly.

(a) $a_n = \dfrac{2n + (-1)^n}{n}$ (b) $a_n = \arctan\left(\dfrac{n+1}{n}\right)$ (c) $a_n = \dfrac{n+1}{1 - \sqrt{n}}$

2011-2012 Fall
MAT123-[Instructor02]-02, [Instructor05]-05 Midterm I
(15/11/2012)
Time: 13:00 - 15:00
Duration: 120 minutes

1. Evaluate the limits, if they exist, and explain your answer. Don't use L'Hôpital's rule.

(a) $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{\sqrt{x}-1}$ (b) $\lim\limits_{x\to 3} \dfrac{\sin(x^2-9)}{x-3}$ (c) $\lim\limits_{x\to -\infty} \left(\sqrt{x^2-x+1}-\sqrt{x^2-2x}\right)$

2. Find the derivatives of the following functions.

(a) $f(x) = \tan^3\left(4\sin^2(3x)\right)$ (b) $f(x) = \left(\cos x^2\right)^x$

(c) Find $f'(0)$ of $f(x) = \ln\left(\dfrac{3^x}{3^x+1}\right)$

3. Evaluate $\lim\limits_{x\to 0} (\mathrm{e}^x - x)^{\frac{1}{x}}$.

4.

(a) Let $F(x)$ be a one-to-one function with inverse F^{-1}. Define a new function

$$p(x) = 1 - 2F\left(\frac{x}{3}\right)$$

Find a formula for p^{-1} in terms of F^{-1}.

(b) Find the derivative of the inverse of the function $f(x) = \arctan x + \mathrm{e}^{123x}$ at $x = 1$. That is, find $\left(f^{-1}\right)'(1)$.

5. Find the equation of the tangent line to the curve $x^2y^2 - 36x = 37$ at $(-1, 1)$.

6. The length of a hypotenuse of a right triangle is constant at 5 cm, and the length of one of its sides is decreasing at a rate of 2 cm/sec. Find the rate of change of the area of the triangle when this side is 3 cm long.

2012-2013 Fall
MAT123-[Instructor02]-02, [Instructor05]-05 Midterm II
(19/12/2012)
Time: 09:00 - 11:00
Duration: 120 minutes

1. A 10 m long wire is cut into two pieces. One piece is bent into an equilateral triangle and the other piece is bent into a circle. If the sum of the areas enclosed by each part is minimum, what is the length of each piece?

2. Integrate the following functions, and write each step in detail.

(a) $\displaystyle\int \frac{dy}{\sqrt{y}\left(1+\sqrt{y}\right)^2}$ (b) $\displaystyle\int_{\pi/6}^{\pi/4} \frac{\cot x}{\ln(\sin x)}\,dx$ (c) $\displaystyle\int \frac{x^3-4x-10}{x^2-x-6}\,dx$

(d) $\displaystyle\int \tan x \sec^{123} x\,dx$ (e) $\displaystyle\int \sqrt{t-t^2}\,dt$ (f) $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx$

3. Integrate the following functions, and write each step in detail.

(a) $\displaystyle\int_0^2 \frac{dx}{\sqrt{|x-1|}}$ (b) $\displaystyle\int_{-\infty}^{\infty} x^2 \mathrm{e}^{-x^3}\,dx$

4. By using the Washer Method, find the volume of the solid generated by revolving the region bounded by the curves $x=y^3$ and $y+x^2=0$ about the x-axis.

2015-2016 Fall Semester
MAT123-07 Midterm
(10/12/2015)

1. Find all local extrema and inflection points of the function $f(x) = \frac{1}{x} + \frac{1}{x^2}$. On which intervals is the function increasing, decreasing, concave upward, or concave downward? Find all asymptotes. Graph the function.

2. Use Rolle's theorem to show that $3\tan x + x^3 = 2$ has exactly one solution on the interval $[0, \pi/4]$.

3. Find the tangent line to the graph of the equation $x\sin(xy - y^2) = x^2 - 1$, at $(1, 1)$.

4. Evaluate the limits.

(a) $\lim\limits_{x\to 0^+} (1 + \sin x)^{\frac{1}{x}}$

(b) $\lim\limits_{x\to 3} \dfrac{x^2 - \lfloor x^2 \rfloor}{x - 3}$

5. A point P is moving in the xy-plane. When P is at $(4, 3)$, its distance to the origin is increasing at a rate of $\sqrt{2}$ cm/s, and its distance to the point $(7, 0)$ is decreasing at a rate of 3 cm/s. Determine the rate of change of the x-coordinate of P at that moment.

6. Sketch the region bounded by $y = 2|x|$ and $y = 8 - x^2$. Find the area of the region.

2016-2017 Fall
MAT123-07 Midterm
(05.12.2016)

1. Evaluate the following limits.

   (a) $\lim_{x\to\infty} [\mathrm{e}^x + 1]^{\frac{1}{x}}$

   (b) $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$

2. For which values of a is

$$f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$$

continuous at every x?

3. Find an equation for the tangent line to the curve $y = x^{\sin x}$, at the point $(\frac{\pi}{2}, \frac{\pi}{2})$.

4. The length of a rectangle decreases at 3 cm/s, while its width increases at 2 cm/s. At a certain moment, the rectangle is 50 cm long and 20 cm wide. Is the area of the rectangle increasing or decreasing, and how fast?

5. Sketch the graph of $y = \dfrac{2x^2}{x^2-1}$.

6. Show that the equation $x^4 + 3x + 1$ has exactly one solution in the interval $[-2, -1]$.

2017-2018 Summer
MAT123-07 Midterm
(24/07/2018)

1. Evaluate

$$\lim_{x \to 3^+} (x-3)^{\ln(x-2)}$$

2. Find constants a and b such that $f(2) + 3 = f(0)$ and $f(x)$ is continuous at $x = 1$ where $f(x)$ is defined by

$$f(x) = \begin{cases} ax + b, & \text{if } x > 1 \\ 3, & \text{if } x = 1 \\ x^2 - 4x + b + 3, & \text{if } x < 1 \end{cases}$$

and $f(x)$ is continuous everywhere.

3. The top of a ladder leaning against a wall slides down the wall at the rate of 6 m/s. When the bottom of the ladder is 9 m from the wall, it is moved away from the wall at the rate of 5 m/s. How long is the ladder?

4. Sketch the graph of

$$f(x) = \frac{x^2 + 1}{x}$$

5. Evaluate the following integrals.

(a) $\int \sqrt{4 - \sqrt{x}}\, dx$

(b) $\int (\ln x)^2\, dx$

6. Let us consider the area A of the region bounded by the curve $x = y^2 - 6y$ and the straight line $x = -y$. Write an integral (but don't evaluate) corresponding to the area A

(i) with respect to the y-axis and

(ii) with respect to the x-axis.

2019-2020 Fall
MAT123 Midterm
(04/11/2019)

1. Evaluate

$$\lim_{x \to 3^+} \cos(x-3)^{\ln\left(\frac{2x}{3}-2\right)}$$

2. Find constants a and b such that $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{\tan ax}{\tan bx}, & \text{if } x < 0 \\ 4, & \text{if } x = 0 \\ ax + b, & \text{if } x > 0 \end{cases}$$

will be continuous at the point $x = 0$.

3. Use the Intermediate Value Theorem to show that the equation $1 - 2x = \sin x$ has at least one real solution. Then use Rolle's Theorem to show it has no more than one solution.

4. Ship A is 60 miles north of point O and moving in the north direction at 20 miles per hour. Ship B is 80 miles east of point O and moving west at 25 miles per hour. How fast is the distance between the ships changing at this moment?

5. Sketch the graph of

$$f(x) = \frac{e^x}{x}$$

6. Evaluate the following integrals.

(a) $\displaystyle\int x^2\sqrt{9+x^2}\,dx$

(b) $\displaystyle\int \tan x \cdot \sec^6 x\,dx$

(c) $\displaystyle\int_4^8 \frac{1}{(x-4)^3}\,dx$

(d) $\displaystyle\int e^{2x} \sin e^x\,dx$

(e) $\displaystyle\int \frac{dx}{\sin x - \cos x}$

7. Let us consider the area A of the region bounded by the curves $x = e^y$, $x = y^2 - 2$ and the straight lines $y = 1$, $y = -1$. Write an integral (but don't evaluate) corresponding to the area A

(i) with respect to the y and
(ii) with respect to the x.

2020-2021 Fall
MAT123 Midterm
(30/11/2020)

1. Evaluate $\lim_{x\to 0^+} (\sqrt{x})^{\ln(x+1)}$.

2. Show that the function $f(x)$ defined by

$$f(x) = \begin{cases} x \arctan \dfrac{1}{x}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ \dfrac{x - \cos x}{x^2}, & \text{if } x < 0 \end{cases}$$

is not continuous at the point $x = 0$.

3. Find an equation of the line which is tangent to the curve

$$\cos y^2 + xy + 1 = 0$$

at the point $\left(\sqrt{\frac{2}{\pi}}, -\sqrt{\frac{\pi}{2}}\right)$. Note that $y = f(x)$.

4. A block of ice in the shape of a cube originally having volume of 3000 cm^3. When it is melting, the surface area is decreasing at the rate of 36 cm^2/h. At what rate does the length of each of its edges decrease at the time its volume is 216 cm^3? Assume that during melting, the block of ice maintains its cubical shape.

5.

(a) Using the Intermediate Value Theorem and Rolle's theorem, show that the equation $e^x + x = 0$ has only one root (Note that if this root c_1, then $c_1 \in (-1, 0)$).

(b) Determine the interval of increase, decrease, and concavity of the function $f(x) = \mathrm{e}^x + x$. By constructing a table, sketch the graph.

6. Determine (but do not evaluate) the integral corresponding to the area of the region bounded by the curves $y = -x^2 + 1$ and $y = |x| - 1$.

7. Evaluate $\int x \ln x \, dx$.

2021-2022 Fall
MAT123-02,05 Midterm
(08/12/2021)

1. Evaluate the following limits or show they do not exist without using L'Hôpital's rule.

(a) $\lim\limits_{x\to-2}\dfrac{|x+2|}{|x|-2}$ (b) $\lim\limits_{x\to0}\dfrac{x^2\sin(1/x)}{\sin(x^2)}$ (c) $\lim\limits_{x\to\frac{1}{2}}\dfrac{\arccos x-\frac{\pi}{3}}{x-\frac{1}{2}}$ $\lim\limits_{x\to-\infty} x+\sqrt{x^2-x-4}$

2. Find the dimensions of the cylinder of largest volume that can be inscribed in the hemisphere of radius 3 if

(a) the cylinder is in the vertical position.

(b) the cylinder is in the horizontal position.

3. A piston is compressing a gas contained in a right circular cylinder. Given the piston is moving into the cylinder at 7 cm/s and the diameter of the cylinder is 10 cm, at what rate is the volume of the gas changing?

4.

(a) Find the limit $\lim\limits_{x\to0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$.

(b) Find an equation of the tangent line to the curve given implicitly by $\dfrac{x}{y}=\cos(\pi xy)$ at the point $(-1,1)$.

5. Sketch the graph of the function $f(x)=\dfrac{2x^2}{(x+1)^2}$.

6. Using IVT and MVT, show that the polynomial function $g(x)=x^7+x^3+x+1$ has only one root.

2022-2023 Fall
MAT123-02,05 Midterm
(23/11/2022)

1. Evaluate the following limits without using L'Hôpital's rule.

(a) $\lim_{x\to 0} \frac{|\sin x|}{x}$ (b) $\lim_{x\to 0^+} xe^{\cos(1/x)}$ (c) $\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1-\sin x}}{x^3}$

2. A balloon is released at point A rises vertically with a constant speed of 5 m/s. Point B is level with and 100 m distant from the point A. How fast is the angle of elevation of the balloon at B changing when the balloon is 200 m above A?

3.

(a) State IVT and MVT.

(b) By IVT and MVT, show that the equation $x^{123}+2x^{85}+3x^{17}+4x-1=0$ has exactly one solution.

4. Find the tangent line to the curve defined implicitly by the equation

$$\sin\left(y^2 e^{2x}\right)+\sqrt{\pi}y = x^2+\pi$$

at $(0, \sqrt{\pi})$.

5. Evaluate $\lim_{x\to\infty} \left(\frac{\ln x}{x}\right)^{1/x}$.

6. Sketch the graph of $f(x) = \frac{x^2-2}{(x-1)^2}$.

2023-2024 Fall
MAT123-02,05 Midterm
(01/11/2023)

1. Evaluate the following limits without using L'Hôpital's rule.

(a) $\lim_{t\to 0} \dfrac{\tan^{-1} t}{\sin^{-1} t}$ (b) $\lim_{x\to 0^+} x\mathrm{e}^{\sin(1/x)}$

2. A spherical balloon is being filled with air in such a way that its radius is increasing at the constant rate of 2 cm/s. At what rate is the volume of the balloon increasing at the instant when its surface has area 4π cm^2?

3.

(a) State MVT.

(b) Using MVT, show that for all $x > 0$,

$$1 + x < \mathrm{e}^x < 1 + x\mathrm{e}^x$$

4. Find the tangent line to the curve defined implicitly by the equation

$$y^2 x^x + xy = 2$$

at the point $(1, 1)$. Note that $y = f(x)$.

5. Find the number A so that $\lim_{x\to\infty} \left(\dfrac{x + A}{x - 2A}\right)^x = 5$.

6. Sketch the graph of $f(x) = (\ln x)^2$.

2024-2025 Fall
MAT123 Midterm
(02/12/2024)

1. Let

$$f(x) = \begin{cases} \dfrac{\tan ax}{\tan bx}, & \text{if } x < 0 \\ 4, & \text{if } x = 0 \\ ax + b, & \text{if } x > 0 \end{cases}$$

Determine the values of a and b such that f is continuous at the point $x = 0$.

2. Use differential to approximate $3\sqrt[3]{66} + 2\sqrt{66}$.

3.

(a) Without using L'Hôpital's rule, evaluate $\lim_{x\to 0} \dfrac{5 - 6\cos x + \cos^2 x}{x \sin x}$.

(b) Prove that $\lim_{x\to -3} \sqrt{-x-2} = 1$ by using the formal definition of limit.

(c) Evaluate $\lim_{x\to 1^+} \left(\sqrt{x}\right)^{\ln(x-1)}$.

4. Coffee is draining out of a conical filter at a rate of 2.25 in.^{3}/min. If the cone is 5 in. tall and has a radius of 2 in., how fast is the coffee level dropping when the coffee is 3 in. deep?

5. Using the Mean Value Theorem, show that $\ln(x+1) < x$ for $x > 0$.

6. Let $f(x) = \dfrac{x^2 - 2}{(x-1)^2}$.

(a) Determine the interval of increase, decrease and concavity of f.

(b) Construct a table.

(c) Sketch the graph of f.

FINALS

2011-2012 Spring
MAT123-[Instructor] Final
(29/05/2012)
Time: 15:00 - 16:50
Duration: 110 minutes

1. Find the length of the curve $y^2 = 4(x+1)^3$ for $0 \leq x \leq 1,\ y > 0$.

2. Given $f(x) = x + 2x^2 + x^3$, find $\left(f^{-1}\right)'(4)$.

3. Evaluate the following integrals.

(a) $\displaystyle\int \cos^3 x \sin^2 x\, dx$ (b) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\, dx$ (c) $\displaystyle\int \frac{x^3-1}{x^3-x}\, dx$ (d) $\displaystyle\int x^{123} \ln x\, dx$

4. Evaluate $\displaystyle\lim_{x\to\infty} (\ln x)^{\frac{1}{x}}$.

5.

(a) Determine the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{3^n}$ converges or diverges. Give reasons for your answer.

(b) Determine the series $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{5^n}\right)$ converges or diverges. Give reasons for your answer.

(c) Use the Ratio Test to determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{\mathrm{e}^{2n}}$ converges or diverges.

(d) Use the Integral Test to determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\mathrm{e}^{n^2}}$ converges or diverges. Be sure to check that the conditions of the integral test are satisfied.

(Bonus)

(a) Geometrically, what does $f'(x)$ mean?

(b) If $f(t)$ describes the displacement of an object in time t, what is $f'(t)$?

2012-2013 Fall
MAT123-[Instructor02]-02, [Instructor05]-05, Final
(23/01/2013)

1. Determine whether each given sequence with nth term converges or diverges. Evaluate the limit of each convergent sequence. Explain all your work, and write clearly.

(a) $a_n = (-1)^n \, n \sin\left(\dfrac{1}{n}\right)$ (b) $a_n = \mathrm{e}^{\cos\left(\frac{1}{n}\right)}$

2. Determine whether the following series converge or diverge. Give reasons for your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(3+n^2)^{3/4}}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \ln\left(\frac{n+1}{n-1}\right)$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{5^n\,(n!)^2}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}\mathrm{e}^{1/n}$

3. Integrate the following functions and write each step in detail.

(a) $\displaystyle\int \frac{dx}{\mathrm{e}^x+1}$ (b) $\displaystyle\int x \arcsin x \, dx$

4. Find the length of the curve $y = \displaystyle\int_0^x \sqrt{\cos(4t)}\, dt$ for $0 \le x \le \pi/8$.

5. For the function $f(x) = \dfrac{x}{x^2-4}$,

(a) Find all the asymptotes of f.

(b) Find the intervals of increase or decrease.

(c) Find the local maximum and minimum values, if any.

(d) Find the intervals of concavity and the inflection points, if any.

(e) Sketch the graph of f.

2014-2015 Fall
MAT123 Makeup
(28/01/2015)

1. Sketch the graph of the function given by $f(x) = \ln(4 - x^2)$.

2. Evaluate the limit $\lim\limits_{x\to\infty} x \arctan \dfrac{1}{x}$.

3. A manufacturer needs to make a cylindrical can that will hold 1500 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

4. Compute the arc length of the graph of the given function $f(x) = \ln(\cos x)$ on the interval $(0, \pi/4]$.

5. Use the Shell Method and then the Washer Method to set up an integral (but do not evaluate) the volume of the solid generated by revolving the region bounded by the curve $y = \sqrt{x}$ and the lines $y = 2 - x$ and $y = 0$ around

(a) the x-axis, and

(b) the y-axis,

respectively.

6. The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the generated surface.

7. Using the Integral Test, determine whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n^2}$$

converges or diverges.

8. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n10^n} (x-2)^n.$$

2014-2015 Summer
MAT123 Final
(23/08/2015)

1. Sketch the graph of $f(x) = \ln(x^2 + 1)$.

2. Determine the area of the region bounded by $y = 4x + 3$ and $y = 6 - x - 2x^2$.

3. Use the Washer Method to find the volume of the solid obtained by revolving the region bounded by $y = 2\sqrt{x-1}$, $y = x - 1$ about the line $x = -1$, respectively.

4. Use the Cylindrical Shell Method to find the volume of the solid obtained by revolving the region bounded by $x = y^2 - 4$ and $x = 6 - 3y$ about $y = -8$.

5. Evaluate the following integrals.

(a) $\displaystyle\int 4\left(\frac{1}{x} - \mathrm{e}^{-x}\right)\cos\left(\mathrm{e}^{-x} + \ln x\right)\,dx$

(b) $\displaystyle\int \frac{x^3 + 10x^2 + 3x + 36}{(x-1)^2\,(x^2+4)^2}\,dx$

(c) $\displaystyle\int \frac{\sqrt{25x^2 - 4}}{x}\,dx$

(d) $\displaystyle\int x^2\cos(4x)\,dx$

(e) $\displaystyle\int \frac{dx}{2x^2 - 3x + 2}$

6. Investigate the convergence of the improper integral $\displaystyle\int_{-\infty}^{0}(1 + 2x)\,\mathrm{e}^{-x}\,dx$.

7. Evaluate the arc length $x = \dfrac{2}{3}(y-1)^{3/2}$ for $1 \le y \le 2$.

2015-2016 Fall
MAT123 Final
(12/01/2016)

1. Find the area of the region bounded by the circle $x^2 + y^2 = 1$ and the graph of the absolute value function $y = |x|$.

2. Use the Washer Method to find the volume of the solid obtained by revolving the region bounded by the parabola $y = 2x^2 - 3$ and the curves $y = -3$, $x = 2$ about the line $y = 7$.

3. Use the Cylindrical Shell Method to find the volume of the solid obtained by revolving the region bounded by $y = x^2$ and $y = -x + 1$ about the line $x = -1$.

4. Evaluate the following integrals.

(a) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

(b) $\displaystyle\int \frac{x^5 + x^4 - 8x^3 + 10x^2 + 12x}{x^2 - 3x + 2}\,dx$

(c) $\displaystyle\int \arccos x\,dx$

(d) $\displaystyle\int \frac{1}{x^2 + 3x + 1}\,dx$

(e) $\displaystyle\int \frac{\sin x}{1 + \sin x}\,dx$

5. Find the surface area of the revolution by rotating the curve $y = \mathrm{e}^x$, $0 \le x \le 1$ about the x-axis.

6. Investigate the convergence of the improper integral $\displaystyle\int_0^{-\infty} \frac{\mathrm{e}^{-x}}{1 + \mathrm{e}^{-x}}\,dx$.

7. Using the Monotone Convergence Theorem, investigate the convergence of the sequence $\displaystyle\left(\frac{\ln n}{n}\right)_{n\in\mathbb{N}}$.

2017-2018 Summer
MAT123-02 Final
(13/08/2018)

1. Let us consider the area A of the region bounded by the curves $y = \mathrm{e}^x$, $y = x^2 - 1$ and the straight lines $x = -1$, $x = 1$. Write an integral (but do not evaluate) corresponding to the area A with respect to the y-axis.

2. Sketch the graph of

$$f(x) = \frac{4x}{x^2 + 1}.$$

3. Evaluate the following integrals.

(a) $\displaystyle\int \frac{\tan x}{\sec^4 x}\, dx$

(b) $\displaystyle\int \frac{\ln x}{x^3}\, dx$

(c) $\displaystyle\int \frac{1}{1 + \sin x}\, dx$

(d) $\displaystyle\int \frac{x + 7}{x^2 (x + 2)}\, dx$

4. Use the Shell Method to determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$, $y = x$ about the line $y = 4$.

5. Find the area of the surface generated by rotating the curve $y = \mathrm{e}^x$, $0 \leq x \leq 1$ about the x-axis.

6. Use the Integral Test to investigate the convergence of the series $\displaystyle\sum_{k=0}^{\infty} k\mathrm{e}^{-k}$.

7. Find the Maclaurin series of the function $f(x) = \ln(1 + x)$.

8. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x - 2)^n}{n \cdot 5^n}$.

2020-2021 Fall
MAT123 Final
(18/01/2021)

1. Consider the region R bounded by the curve $y = x^3$, and the straight lines $y = -x$ and $y = x + 6$.

(i) Write down the integral corresponding to the area of R with respect to x.

(ii) Write down the integral corresponding to the area of R with respect to y.

2. Consider the solid S obtained by revolving the region in the first quadrant bounded by $x = -y^2 + 1$, $y^2 = x$ and $y = 1/2$ about $x = -1$.

(i) Using the Shell Method, write down the integral corresponding to the volume of the solid S.

(ii) Using the Washer Method, write down the integral corresponding to the volume of the solid S.

3. Evaluate the following integrals.

(a) $\displaystyle\int \frac{dx}{x^{2/3}\left(\sqrt[3]{x}+4\right)}$

(b) $\displaystyle\int (\ln x)^2\, dx$

(c) $\displaystyle\int \frac{dx}{\sqrt{3+x^2}}$

(d) $\displaystyle\int \frac{dx}{2+\sin x}$

4. Using the Monotone Convergence Theorem, show that the sequence $\left(\dfrac{n^2+1}{n^3}\right)_{n\in\mathbb{N}}$ is convergent.

5. Use the Integral Test to determine whether the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4+k^2}$ converges or diverges.

6. Find the convergence set for the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(x+1)^k}{2k}$.

7. Using a Maclaurin series, show that $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$.

2021-2022 Fall
MAT123-02,05 Final
(07/01/2022)

1.

(a) Find $\displaystyle\int \frac{dx}{x^3 - 4x^2 + 3x}$.

(b) Evaluate $\displaystyle\lim_{x \to \frac{\pi}{2}} \frac{\displaystyle\int_{\pi/2}^{x} \ln(\sin t)\, dt}{\sin x - 1}$.

(c) Evaluate the improper integral $\displaystyle\int_1^2 \frac{dx}{(x-1)^{2/3}}$.

2. Consider the region R bounded by the curves $y = \arctan x$, $y = \ln x$ and the lines $x = \dfrac{1}{\sqrt{3}}$ and $x = 1$.

(a) Find the area of the region R.

(b) Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of a solid obtained by rotating the region R about the y-axis.

(c) Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of a solid obtained by rotating the region R about the line $y = 2$.

3. Determine whether each series is convergent or divergent. Explain your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} (\arctan n - \arctan(n-1))$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{n^2}{1+n^4}\right)$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

2022-2023 Fall
MAT123-02,05 Final
(11/01/2023)

1. The radius R of a spherical ball is measured as 14 in.

(a) Use differentials to estimate the maximum propagated error in computing the volume V if R is measured with a maximum error of 1/8 inches.

(b) With what accuracy must the radius R be measured to guarantee and error of at most 2 in^3 in the calculated volume?

2. Evaluate the following integrals.

(a) $\displaystyle\int \frac{dx}{\sqrt{x}\,(\sqrt{x}+2)}$

(b) $\cancel{\displaystyle\int \frac{\sin x}{x^2+1}\,dx}$

(c) $\displaystyle\int \frac{dx}{\sqrt{3-x^2}}$

(d) $\displaystyle\int \frac{dx}{2+\cos x}$

3. Use the Shell Method and then the Washer Method to set up an integral (but do not evaluate) the volume of the solid generated by revolving the region R about the y-axis, where R is bounded by the curve $y = x^2$ and the line $y = -x + 1$.

4. Find the area of the surface obtained by rotating the arc of the curve $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on the interval $[1/2, 1]$ about the x-axis.

5. Using the Integral Test, determine whether the series

$$\sum_{n=1}^{\infty} \frac{2}{3n+5}$$

converges or diverges.

6. Find the Maclaurin series for $f(x) = \dfrac{1}{x^2 - 5x + 6}$.

2022-2023 Fall
MAT123-02,05 Final
(13/01/2023)

1. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$ (b) $\displaystyle\int_0^{\pi/4} x\sec^2 x\,dx$

2.

(a) Consider the finite region between the curves $y = \mathrm{e}^{-x}$, $y = x/\mathrm{e}$, and the y-axis. Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of the solid obtained by rotating this region about the x-axis.

(b) Consider the infinite region between the curves $y = \mathrm{e}^{-x}$, $y = x/\mathrm{e}$ and the x-axis. Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of the solid obtained by rotating this region about the line $y = -1$.

(c) Evaluate the area of the region given in part (a).

3. Determine whether each series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\arctan(n^2)}$ (b) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^3}\right)$ (c) $\displaystyle\sum_{n=1}^{\infty} n\mathrm{e}^{-n^2}$

4.

(a) Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^{n+1}\cdot(x+1)^n}{n\cdot 3^n}.$$

(b) Using the formula $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, find the Maclaurin series of the function $f(x) = \dfrac{x^{123}}{1+x^4}$.

2024-2025 Fall
MAT123-02,05 Final
(09/01/2025)

1. Evaluate the following integrals.

(a) $\displaystyle\int \frac{dx}{2\cos x + 3}$

(b) $\displaystyle\int \frac{\sqrt{5-x^2}}{x}\,dx$

(c) $\displaystyle\int \frac{dx}{1-x^4}$

(d) $\displaystyle\int \ln\left(1+x^2\right)\,dx$

2. Consider the region R bounded by the curves $y = x$, $y = x^2$ and $y = x^2/2$.

(a) Find the area of the region R.

(b) Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of a solid obtained by rotating the region R about the y-axis.

(c) Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of the solid obtained by rotating the region R about the line $y = 2$.

3. Use the Integral Test to determine the existence of the sum of the series $\displaystyle\sum_{n=1}^{\infty} n\mathrm{e}^{-n^2}$.

4. Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\right)^{1/n}$ is convergent or divergent.

5. Find the Maclaurin series of $f(x) = \sqrt{\mathrm{e}^x}$ and determine the interval of convergence of this series.

2024-2025 Fall
MAT123-02,05 Makeup
(31/01/2025)

1.

(a) Find $\displaystyle\int \frac{\sin^3 x}{\sqrt{\cos x}}\, dx$.

(b) Find $\displaystyle\int \frac{dx}{x^3 - 4x^2 + 3x}$.

(c) Evaluate $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{\int_{\pi/2}^{x} \ln(\sin t)\, dt}{\sin x - 1}$.

(d) Evaluate the improper integral $\displaystyle\int_0^2 \frac{dx}{(x-1)^{2/3}}$.

2. Consider the region R bounded by the curves $y = \arctan x$, $y = \ln x$ and the lines $x = \dfrac{1}{\sqrt{3}}$ and $x = 1$.

(a) Sketch the region and find the area of the R.

(b) Write a definite integral (do not evaluate) by using the Washer Method, which gives the volume of a solid obtained by rotating the region R about the y-axis.

(c) Write a definite integral (do not evaluate) by using the Cylindrical Shell Method, which gives the volume of a solid obtained by rotating the region R about the line $y = 2$.

3. Determine whether each series is convergent or divergent. Explain your answer.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

4. Find the Taylor series of the function $f(x) = \ln x$ at $c = 1$ and determine the interval of convergence.

MIDTERM SOLUTIONS

2011-2012 Spring Midterm I (06/04/2012) Solutions
(Last update: 29/08/2025 19:24)

1. y is implicitly defined as a function of x. Differentiate each side.

$$\frac{d}{dx}\left(x^2+2xy+y^2\right)=\frac{d}{dx}(4)$$

$$2x+2y\cdot 1+2x\cdot\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}(2x+2y)=-2x-2y$$

$$\frac{dy}{dx}=-1$$

Recall the equation of a straight line: $y-y_0=m(x-x_0)$, where m is simply $\frac{dy}{dx}$. Therefore, the tangent line at $(3,-1)$ is as follows.

$$\boxed{y+1=-1(x-3)}$$

Furthermore, since $\frac{dy}{dx}=-1$, the tangent line consists of every point of the upper line. In other words, the tangent line is the upper line itself. The equation $x^2+2xy+y^2=4$ forms two parallel straight lines in the $xy-$coordinate system.

2. Let $(x,\sqrt{x})$ be a point on this parabola. The distance between the points can be expressed using the Pythagorean theorem as follows.

$$f(x)=(2-x)^2+\left(\sqrt{x}-0\right)^2=L^2$$

Take the derivative of both sides and set $f'(x)=0$ to find the critical points.

$$f'(x)=2(2-x)\cdot(-1)+1=2x-3=0 \implies x=\frac{3}{2}$$

Now, verify whether this is a local minimum by taking the second derivative.

$$f''(x)=(2x-3)'=2>0$$

Since this is a local minimum of f, the distance is closest at $x=\frac{3}{2}$. The point we are looking for is

$$\boxed{\left(\frac{3}{2},\sqrt{\frac{3}{2}}\right)}$$

3.

(a) The inequality $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ holds for all $x \in \mathbb{R}$ except for $x = 0$. For small x, we can multiply each side of the inequality by $\sqrt{x^2 + 2x^3}$. Using the squeeze theorem, the limit is equal to 0.

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$-\sqrt{x^2 + 2x^3} \leq \sqrt{x^2 + 2x^3}\sin\left(\frac{1}{x}\right) \leq \sqrt{x^2 + 2x^3}$$

$$\lim_{x\to 0} -\sqrt{x^2 + 2x^3} = \lim_{x\to 0} \sqrt{x^2 + 2x^3} = 0 \implies \lim_{x\to 0} \sqrt{x^2 + 2x^3}\sin\left(\frac{1}{x}\right) = \boxed{0}$$

(b) Factor each side of the fraction and eliminate like terms.

$$\lim_{x\to 3} \frac{x^2 - 9}{x^2 - x - 6} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(x+2)} = \lim_{x\to 3} \frac{x+3}{x+2} = \boxed{\frac{6}{5}}$$

(c) The expression is in the form $\infty - \infty$. Expand the expression by multiplying by its conjugate to eliminate the indetermination.

$$\lim_{x\to -\infty} \sqrt{x^2 - 4x} + x = \lim_{x\to -\infty} \left[\left(\sqrt{x^2 - 4x} + x\right) \cdot \frac{\sqrt{x^2 - 4x} - x}{\sqrt{x^2 - 4x} - x}\right] = \lim_{x\to -\infty} \frac{x^2 - 4x - x^2}{\sqrt{x^2 - 4x} - x}$$

$$= \lim_{x\to -\infty} \frac{-4x}{\sqrt{x^2 - 4x} - x} = \lim_{x\to -\infty} \frac{-4x}{|x|\sqrt{1 - \frac{4}{x}} - x}$$

$$= \lim_{x\to -\infty} \frac{-4}{-\sqrt{1 - \frac{4}{x}} - 1} = \lim_{x\to -\infty} \frac{4}{2} = \boxed{2}$$

(d) Recall the definition of the derivative of a function at a point. Let f be a differentiable function, then

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

If we set $f(x) = x^{123}$, then we can differentiate f at $x = 1$.

$$\lim_{h\to 0} \frac{(1+h)^{123} - 1}{h} = f'(1) = 123 \cdot (1)^{122} = \boxed{123}$$

4.

(a) Take the logarithm of each side to compute the derivative easily.

$$f(x) = x^{\cos(x^3)}$$
$$\ln(f(x)) = \ln\left[x^{\cos(x^3)}\right] = \cos\left(x^3\right) \cdot \ln x$$
$$\frac{d}{dx}\left[\ln(f(x))\right] = \frac{d}{dx}\left[\cos\left(x^3\right) \cdot \ln x\right]$$
$$\frac{1}{f(x)} \cdot f'(x) = -\sin\left(x^3\right) \cdot 3x^2 \cdot \ln x + \cos\left(x^3\right) \cdot \frac{1}{x}$$
$$\boxed{f'(x) = x^{\cos(x^3)} \cdot \left[-\sin\left(x^3\right) \cdot 3x^2 \cdot \ln x + \cos\left(x^3\right) \cdot \frac{1}{x}\right]}$$

(b) Apply the chain rule accordingly.

$$f(x) = \tan\left(\mathrm{e}^{2x}\sin(3x)\right)$$
$$\boxed{f'(x) = \sec^2\left(\mathrm{e}^{2x}\sin(3x)\right) \cdot \left[\mathrm{e}^{2x} \cdot 2 \cdot \sin(3x) + \mathrm{e}^{2x} \cdot \cos(3x) \cdot 3\right]}$$

(c) Compute the first and second derivatives, respectively, applying the chain rule and the quotient rule accordingly.

$$f(x) = \ln\left(\frac{x^2}{x^2+4}\right)$$
$$f'(x) = \frac{x^2+4}{x^2} \cdot \frac{2x \cdot (x^2+4) - x^2 \cdot 2x}{(x^2+4)^2} = \frac{8}{x^3+4x}$$
$$f''(x) = -\frac{8}{(x^3+4x)^2} \cdot \left(3x^2+4\right) = \boxed{-\frac{24x^2+32}{(x^3+4x)^2}}$$

5.

(a) Find the horizontal asymptotes.

$$\lim_{x\to\pm\infty} \frac{1}{x^2-4} = 0$$

Find the vertical asymptotes. The expression is undefined for $x = \pm 2$.

$$\lim_{x\to 2^+} \frac{1}{x^2-4} = \lim_{x\to -2^-} \frac{1}{x^2-4} = \infty$$
$$\lim_{x\to 2^-} \frac{1}{x^2-4} = \lim_{x\to -2^+} \frac{1}{x^2-4} = -\infty$$

$$\boxed{\text{The horizontal asymptote is } y = 0. \text{ The vertical asymptotes are } x = \pm 2.}$$

(b) Compute the first derivative and set it to 0 to find the critical points.

$$f'(x) = -\frac{1}{\left(x^2-4\right)^2} \cdot 2x$$

f is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$.

$$\boxed{\text{f is decreasing for } x > 0, \text{increasing for } x < 0.}$$

(c) The *only* critical point occurs at $x = 0$. Compute the second derivative to check whether this is a local minimum or local maximum.

$$f''(x) = -\frac{2 \cdot \left(x^2-4\right)^2 - 2x \cdot 2 \cdot \left(x^2-4\right) \cdot (2x)}{\left(x^2-4\right)^4} = \frac{8+6x^2}{\left(x^2-4\right)^3}$$

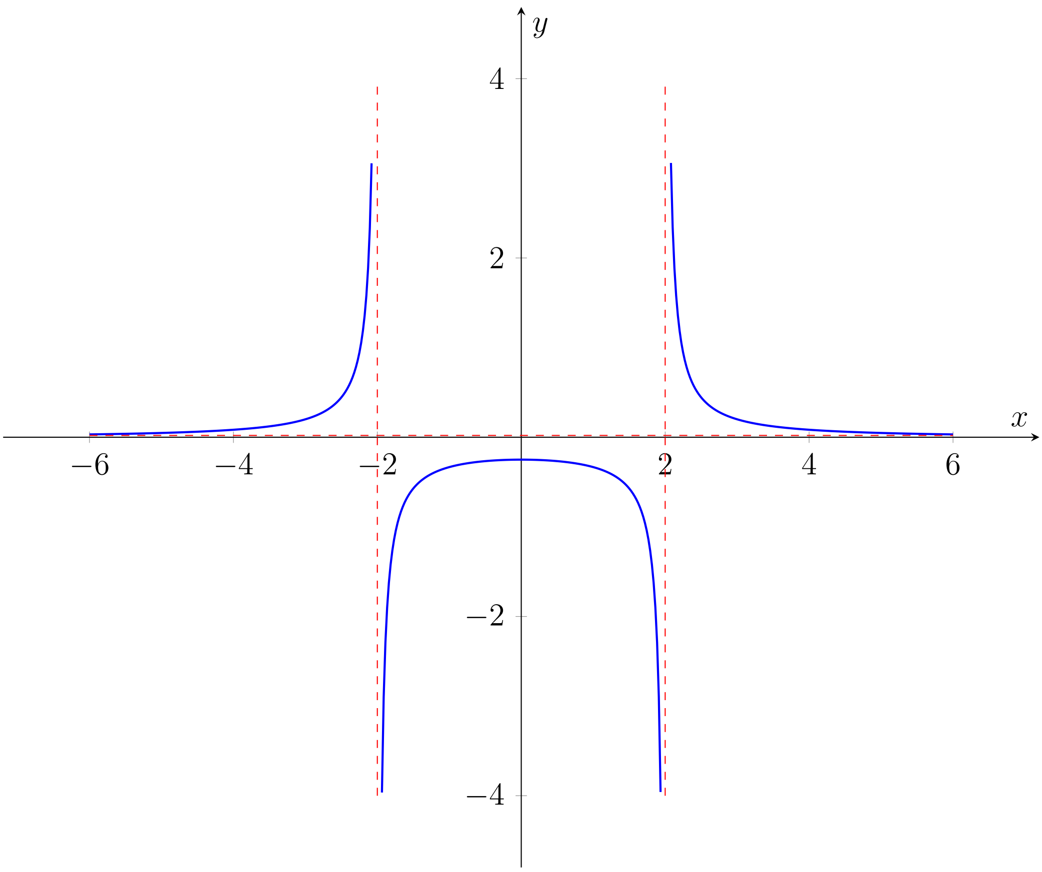
$f''(0) = \dfrac{8+6 \cdot 0^2}{\left(0^2-4\right)^3} = -\dfrac{1}{8} < 0$. Therefore, $(0, f(0))$ is a local maximum.

$$\boxed{\begin{array}{c}\text{No local minimums exist.}\\ \text{The } \textit{only} \text{ local maximum occurs at } x = 0, \text{ which is } \left(0, -\dfrac{1}{4}\right).\end{array}}$$

(d) $8 + 6x^2 \geq 0$. Therefore, no inflection points. f is concave up if $f''(x) > 0$, concave down if $f''(x) < 0$.

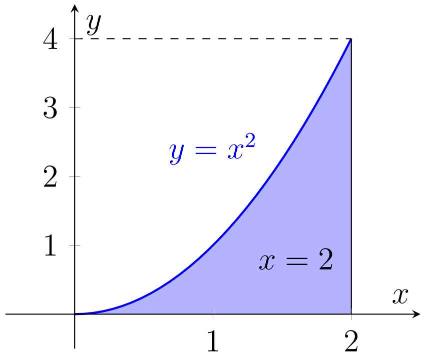
$$\boxed{\begin{array}{c}\text{No inflection points exist.}\\ \text{f is concave up for } x > 2 \text{ and } x < -2. \text{ f is concave down for } |x| < 2.\end{array}}$$

(e)

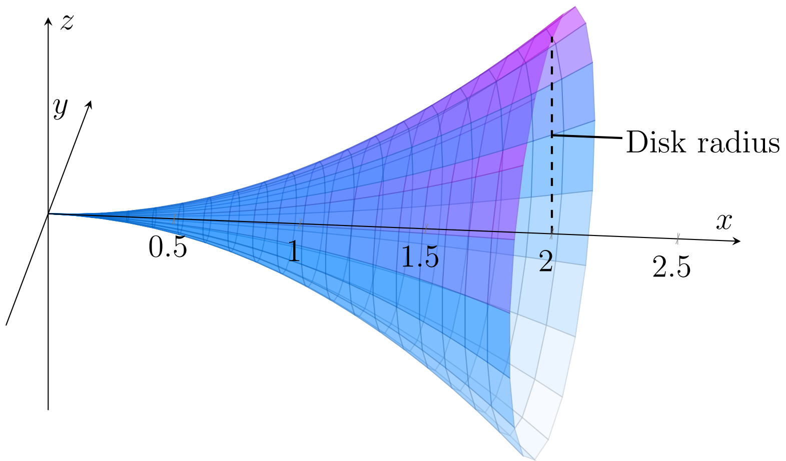


2011-2012 Spring Midterm II (08/05/2012) Solutions
(Last update: 29/08/2025 20:11)

1.

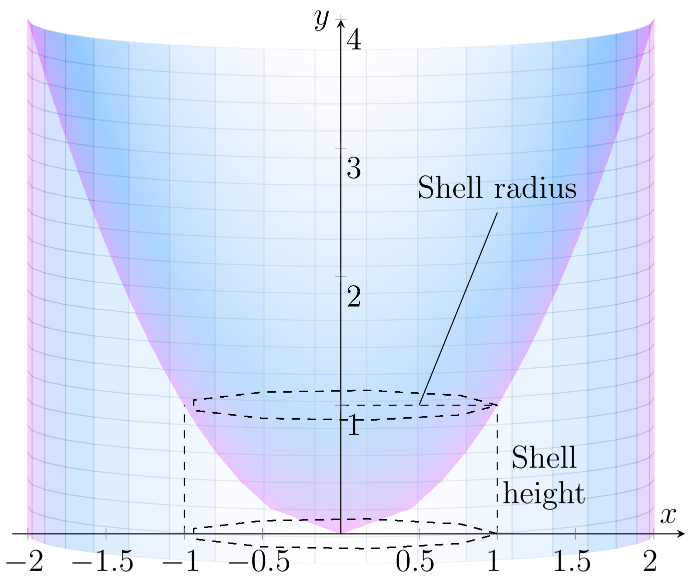


(a)



$$\text{Volume} = \int_{\mathcal{D}} \pi \cdot (r(x))^2 \, dx = \int_0^2 \pi \left(x^2\right)^2 \, dx = \pi \int_0^2 x^4 \, dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \boxed{\frac{32\pi}{5}}$$

(b)



$$\text{Volume} = \int_{\mathcal{D}} 2\pi \cdot r(x) \cdot h(x)\, dx = 2\pi \int_0^2 x \cdot x^2\, dx = 2\pi \int_0^2 x^3\, dx$$

$$= 2\pi \left[\frac{x^4}{4}\right]_0^2 = 2\pi \left(\frac{2^4}{4} - 0\right) = \boxed{8\pi}$$

2.

(a) The expression $|x^2 - 1|$ is the same as $x^2 - 1$ for $x > 1$ and $1 - x^2$ for $x < 1$. We can write the equivalent expression below.

$$\int_0^3 \left|x^2 - 1\right| = dx \int_0^1 \left(1 - x^2\right)\, dx + \int_1^3 (x^2 - 1)\, dx = \left[x - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - x\right]_1^3$$

$$= \left[\left(1 - \frac{1^3}{3}\right) - 0\right] + \left[\left(\frac{3^3}{3} - 3\right) - \left(\frac{1^3}{3} - 1\right)\right] = \boxed{\frac{22}{3}}$$

(b)

$$\int \frac{1}{x^2 + 2x + 1}\, dx = \int \frac{1}{(x+1)^2}\, dx = \boxed{-\frac{1}{x+1} + c, \quad c \in \mathbb{R}}$$

(c)

$$\int \frac{1}{x^2 + 2x + 2}\, dx = \int \frac{1}{(x+1)^2 + 1}\, dx \quad [u = x + 1 \implies du = dx]$$

$$= \int \frac{1}{u^2 + 1}\, du = \arctan(u) + c = \boxed{\arctan(x+1) + c, \quad c \in \mathbb{R}}$$

(d) We can use the method of partial fraction decomposition.

$$\int \frac{1}{x^2 + 3x + 2}\, dx = \int \frac{1}{(x+1)(x+2)}\, dx = \int \left(\frac{A}{x+1} + \frac{B}{x+2}\right)\, dx \tag{1}$$

$$A(x+2) + B(x+1) = 1$$
$$x(A+B) + 2A + B = 1$$
$$\therefore A + B = 0 \quad [\text{eliminate } x] \to 2A + B = 1$$

$$\left.\begin{array}{r} A + B = 0 \\ 2A + B = 1 \end{array}\right\} \quad A = 1, \quad B = -1$$

Plug the values of A and B into (1).

$$\int \left(\frac{A}{x+1} + \frac{B}{x+2}\right)\, dx = \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right)\, dx = \boxed{\ln|x+1| - \ln|x+2| + c, \quad c \in \mathbb{R}}$$

(e) Use the trigonometric identity $\cos(2x) = 1 - 2\sin^2 x$.

$$\int_{-\pi/6}^{0} \sqrt{1-\cos(6x)}\,dx = \int_{-\pi/6}^{0} \sqrt{2\sin^2(3x)}\,dx = \sqrt{2}\int_{-\pi/6}^{0} |\sin(3x)|\,dx$$

$\sin x < 0$ for $-\pi < x < \pi$. Therefore, the integral can be rewritten as follows.

$$\sqrt{2}\int_{-\pi/6}^{0} |\sin(3x)|\,dx = \sqrt{2}\int_{-\pi/6}^{0} -\sin(3x)\,dx = \sqrt{2}\left[\frac{1}{3}\cos(3x)\right]_{-\pi/6}^{0}$$

$$= \frac{\sqrt{2}}{3}\left[\cos 0 - \cos\left(-\frac{\pi}{2}\right)\right] = \boxed{\frac{\sqrt{2}}{3}}$$

3. We have an improper integral where the limits are $\pm\infty$. Use limits to handle improper integrals accurately.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{A\to\infty}\int_{-A}^{A} \frac{1}{1+x^2}\,dx = \lim_{A\to\infty} \arctan(x)\Big|_{-A}^{A}$$

$$= \lim_{A\to\infty} (\arctan A - \arctan(-A)) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$$

The value of the improper integral is finite. Therefore, this integral is convergent.

4.

(a) The limit is in the indeterminate form $0/0$. We can apply L'Hôpital's rule to eliminate the indeterminate form.

$$\lim_{x\to 0} \frac{3^x - 1}{5^x - 1} \overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{3^x \cdot \ln 3}{5^x \cdot \ln 5} = \log_5 3 \cdot \lim_{x\to 0} \left(\frac{3}{5}\right)^x = \boxed{\log_5 3}$$

(b) The first method is to evaluate the integral in the limit.

$$\lim_{x\to 0} \frac{\int_0^{x^2} \sin t\,dt}{x} = \lim_{x\to 0} \frac{-\cos t\Big|_{t=0}^{t=x^2}}{x} = \lim_{x\to 0} \frac{-\cos x^2 - (-\cos 0)}{x} = \lim_{x\to 0} \frac{1-\cos x^2}{x}$$

Multiply by the expression inside the limit $\dfrac{x}{x}$. Notice that we obtain a standard form.

$$\lim_{x\to 0}\left(\frac{1-\cos x^2}{x}\cdot\frac{x}{x}\right) = \lim_{x\to 0}\frac{1-\cos x^2}{x^2}\cdot\lim_{x\to 0} x \qquad \left[\lim_{u\to 0}\frac{1-\cos u}{u} = 0\right]$$

$$= 0 \cdot 0 = \boxed{0}$$

The second method is to use L'Hôpital's rule because of the 0/0 form.

$$\lim_{x\to 0} \frac{\displaystyle\int_0^{x^2} \sin t\, dt}{x} \overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{\displaystyle\frac{d}{dx}\int_0^{x^2} \sin t\, dt}{1}$$

Let $u = x^2$, then $du = 2x\, dx$. By the Fundamental Theorem of Calculus, the limit can be rewritten as follows.

$$\lim_{x\to 0} \frac{d}{dx} \int_0^{x^2} \sin t\, dt = \lim_{x\to 0} \frac{d}{du}\left(\int_0^{u} \sin t\, dt\right)\frac{du}{dx} = \lim_{x\to 0} (\sin u \cdot 2x) = \lim_{x\to 0} \left[2x \sin\left(x^2\right)\right] = \boxed{0}$$

5.

(a)

$$a_n = \frac{2n + (-1)^n}{n} = 2 + \frac{(-1)^n}{n}$$

The sequence converges to $\boxed{2}$ because $\dfrac{(-1)^n}{n} \to 0$ as $n \to \infty$.

(b) arctan is continuous everywhere. Therefore, we can take the limit inside the expression.

$$\lim_{n\to\infty} \arctan\left(\frac{n+1}{n}\right) = \arctan \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = \arctan 1 = \frac{\pi}{4}$$

The sequence converges to $\boxed{\dfrac{\pi}{4}}$.

(c)

$$\lim_{n\to\infty} \frac{n+1}{1-\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}\left(\sqrt{n} + \frac{1}{\sqrt{n}}\right)}{\sqrt{n}\left(\frac{1}{\sqrt{n}} - 1\right)} = \lim_{n\to\infty} \frac{\sqrt{n} + \frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}} - 1} = -\infty$$

Notice that the denominator approaches -1 as $n \to \infty$ and the numerator approaches ∞ as $n \to \infty$. The sequence diverges to $\boxed{-\infty}$.

2011-2012 Fall Midterm I (15/11/2012) Solutions
(Last update: 12/10/2025 03:41)

1.

(a) Multiply each side by the conjugate of the denominator to eliminate the indetermination.

$$\lim_{x\to 1}\frac{x^2+x-2}{\sqrt{x}-1}=\lim_{x\to 1}\left[\frac{(x+2)(x-1)}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}\right]=\lim_{x\to 1}\frac{(x+2)(x-1)\left(\sqrt{x}+1\right)}{x-1}$$

$$=\lim_{x\to 1}\left[(x+2)\left(\sqrt{x}+1\right)\right]=3\cdot 2=\boxed{6}$$

(b)

$$\lim_{x\to 3}\frac{\sin(x^2-9)}{x-3}=\lim_{x\to 3}\left[\frac{\sin(x^2-9)}{x-3}\cdot\frac{x+3}{x+3}\right]=\lim_{x\to 3}\left[\frac{\sin(x^2-9)}{x^2-9}\right]\cdot\lim_{x\to 3}(x+3)$$

The value $\lim_{x\to 0}\dfrac{\sin u}{u}$ can be evaluated by using the squeeze theorem, and it could be expected that we knew the value of this limit prior to the examination. Set $u=x^2-9$. So, the left-hand limit is 1.

$$=\lim_{x\to 3}\left[\frac{\sin(x^2-9)}{x^2-9}\right]\cdot\lim_{x\to 3}(x+3)=1\cdot 6=\boxed{6}$$

(c) This is an indeterminate $(\infty-\infty)$ form. We expand the expression by using its conjugate and dividing each side of the fraction by x to eliminate the indeterminate form.

$$\lim_{x\to-\infty}\left(\sqrt{x^2-x+1}-\sqrt{x^2-2x}\right)$$

$$=\lim_{x\to-\infty}\left[\sqrt{x^2-x+1}-\sqrt{x^2-2x}\cdot\frac{\sqrt{x^2-x+1}+\sqrt{x^2-2x}}{\sqrt{x^2-x+1}+\sqrt{x^2-2x}}\right]$$

$$=\lim_{x\to-\infty}\frac{x^2-x+1-(x^2-2x)}{\sqrt{x^2-x+1}+\sqrt{x^2-2x}}=\lim_{x\to-\infty}\left(\frac{x+1}{\sqrt{x^2-x+1}+\sqrt{x^2-2x}}\cdot\frac{x}{x}\right)$$

$$=\lim_{x\to-\infty}\frac{\dfrac{x+1}{x}}{\dfrac{|x|\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+|x|\sqrt{1-\frac{1}{x}}}{x}}=\lim_{x\to-\infty}-\frac{1+\dfrac{1}{x}}{\sqrt{1-\dfrac{1}{x}+\dfrac{1}{x^2}}+\sqrt{1-\dfrac{2}{x}}}\qquad\left[|x|=-x\right]$$

$$=-\frac{1-0}{\sqrt{1+0-0}+\sqrt{1+0}}=\boxed{-\frac{1}{2}}$$

2.

(a) Apply the chain rule accordingly.

$$\boxed{f'(x) = 3\tan^2\left(4\sin^2(3x)\right)\cdot\sec^2\left(4\sin^2(3x)\right)\cdot 8\sin(3x)\cdot\cos(3x)\cdot 3}$$

(b) Take the logarithm of both sides to differentiate easily.

$$\ln\left(f(x)\right) = \ln\left[\left(\cos x^2\right)^x\right] = x\ln\left(\cos x^2\right)$$

$$\frac{d}{dx}\ln\left(f(x)\right) = \frac{d}{dx}\left[x\ln\left(\cos x^2\right)\right]$$

$$\frac{1}{f(x)}\cdot f'(x) = 1\cdot\left[\ln\left(\cos x^2\right)\right] + x\cdot\frac{1}{\cos x^2}\cdot\left(-\sin x^2\right)\cdot 2x$$

$$f'(x) = f(x)\left[\ln\left(\cos x^2\right) - 2x^2\cdot\tan x^2\right]$$

$$\boxed{f'(x) = \left(\cos x^2\right)^x\cdot\left[\ln\left(\cos x^2\right) - 2x^2\cdot\tan x^2\right]}$$

(c) Rewrite the right-hand side using the property of logarithms. Take the first derivative afterwards.

$$f(x) = \ln\left(\frac{3^x}{3^x+1}\right) = \ln\left(3^x\right) - \ln\left(3^x+1\right)$$

$$f'(x) = \frac{1}{3^x}\cdot 3^x\cdot\ln(3) - \frac{1}{3^x+1}\cdot 3^x\cdot\ln(3) = \ln(3)\cdot\left[1 - \frac{3^x}{3^x+1}\right]$$

$$f'(0) = \ln(3)\cdot\left[1 - \frac{3^0}{3^0+1}\right] = \boxed{\frac{\ln 3}{2}}$$

3. Let L be the value of the limit.

$$L = \lim_{x\to 0}\left(e^x - x\right)^{\frac{1}{x}}$$

$$\ln(L) = \ln\left[\lim_{x\to 0}\left(e^x - x\right)^{\frac{1}{x}}\right]$$

The expression is defined for $x \neq 0$. Therefore, we can take the logarithm function inside the limit. After that, apply L'Hôpital's rule to eliminate the indeterminate form.

$$\ln(L) = \lim_{x\to 0}\ln\left[\left(e^x - x\right)^{\frac{1}{x}}\right] = \lim_{x\to 0}\frac{\ln\left(e^x - x\right)}{x} \quad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 0}\frac{\frac{1}{e^x - x}\cdot\left(e^x - 1\right)}{1} = \lim_{x\to 0}\frac{e^x - 1}{e^x - x} = \frac{e^0 - 1}{e^0 - 0} = 0$$

If $\ln(L) = 0$, then $\boxed{L = 1}$.

4.

(a)

$$
\begin{aligned}
p(x) &= 1 - 2F\left(\frac{x}{3}\right) \\
p(x) - 1 &= -2F\left(\frac{x}{3}\right) \\
\frac{1 - p(x)}{2} &= F\left(\frac{x}{3}\right) \\
F^{-1}\left(\frac{1 - p(x)}{2}\right) &= \frac{x}{3} \\
3F^{-1}\left(\frac{1 - p(x)}{2}\right) &= x \\
3F^{-1}\left(\frac{1 - p\left(p^{-1}(x)\right)}{2}\right) &= p^{-1}(x)
\end{aligned}
$$

$$
\boxed{p^{-1}(x) = 3F^{-1}\left(\frac{1 - x}{2}\right)}
$$

(b) Find a root so that $f(x_0) = \arctan(x_0) + \mathrm{e}^{123 \cdot x_0} = 1$. We can intuitively say that the root is small because both $\arctan x$ and e^x are strictly increasing everywhere. Try $x = 0$.

$$
f(0) = \arctan 0 + \mathrm{e}^{123 \cdot 0} = 0 + 1 = 1
$$

Therefore, $f^{-1}(1) = 0$. The derivative of an inverse function at a given point is

$$
\left(f^{-1}\right)'(1) = \frac{1}{f'(f^{-1}(1))}
$$

Calculate $f'(x)$ and $(f^{-1})(1)$.

$$
f'(x) = \frac{1}{1 + x^2} + 123\mathrm{e}^{122x}
$$

$$
\left(f^{-1}\right)'(1) = \frac{1}{f'(0)} = \left(\frac{1}{1 + 0^2} + 123\mathrm{e}^{122 \cdot 0}\right)^{-1} = \boxed{\frac{1}{124}}
$$

5. Consider $y = f(x)$. Differentiate both sides implicitly.

$$
\frac{d}{dx}\left(x^2 y^2 - 36x\right) = \frac{d}{dx} 37
$$

$$
2xy^2 + x^2 \cdot 2y \cdot \frac{dy}{dx} - 36 = 0
$$

$$
x^2 \cdot 2y \cdot \frac{dy}{dx} = 36 - 2xy^2
$$

$$
\frac{dy}{dx} = \frac{36 - 2xy^2}{x^2 \cdot 2y}
$$

Calculate $\frac{dy}{dx}$ at $(-1, 1)$.

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{36 - 2(-1) \cdot 1^2}{(-1)^2 \cdot 2 \cdot 1} = 19 \tag{2}$$

Using the straight line formula, we find the tangent line. Recall: $y - y_0 = m(x - x_0)$, where m can be substituted with (1).

$$\boxed{y - 1 = 19(x + 1)}$$

6. Let $x(t)$, $y(t)$, $l(t)$ represent the lengths of the sides as functions of time. We can set up the following equation for the area of the right triangle.

$$A(t) = \frac{x(t) \cdot y(t)}{2}$$

Take the derivative of both sides.

$$A'(t) = \frac{1}{2}\left(x'(t) \cdot y(t) + x(t) \cdot y'(t)\right) \tag{3}$$

We also know that, by the Pythagorean theorem,

$$l^2(t) = x^2(t) + y^2(t)$$

Take the derivative of both sides.

$$2l(t)l'(t) = 2x(t)x'(t) + 2y(t)y'(t)$$

Since the length of the hypotenuse is constant, $l'(t) = 0$. Therefore,

$$x(t)x'(t) = -y(t)y'(t) \tag{4}$$

At $t = t_0$, we have $l(t_0) = 5$, $x(t_0) = 3$, $x'(t_0) = -2$, and by the Pythagorean theorem, $y(t_0) = \sqrt{5^2 - 3^2} = 4$. Calculate $y'(t_0)$ from (3).

$$y'(t_0) = -\frac{3 \cdot (-2)}{4} = \frac{3}{2}$$

Plug the necessary values into (2) to find the rate of change of the area.

$$A'(t_0) = \frac{1}{2}\left((-2) \cdot 4 + 3 \cdot \frac{3}{2}\right) = \boxed{-\frac{7}{4}\,\mathrm{cm}^2/\mathrm{s}}$$

2012-2013 Fall Midterm II (19/12/2012) Solutions
(Last update: 29/08/2025 19:47)

1. Let x be the length of the wire that is used to form the equilateral triangle. So, $10-x$ is the length of the other piece. The area of the triangle is

$$A_1 = \frac{x^2\sqrt{3}}{4}$$

The perimeter of the circle is $10-x$. Therefore, $2\pi r = 10 - x$, where r is the radius of the circle. The area of the circle can be extracted from the formula $A_2 = \pi r^2$. Solve the former equation for r, and express the area in terms of x.

$$r = \frac{10-x}{2\pi} \to A_2 = \pi\left(\frac{10-x}{2\pi}\right)^2 = \frac{100-20x+x^2}{4\pi}$$

Let $A(x)$ be the function of length representing the sum of the areas.

$$A(x) = \frac{x^2\sqrt{3}}{4} + \frac{100-20x+x^2}{4\pi}$$

Minimize $A(x)$ by taking the first derivative and setting it to 0.

$$A'(x) = \frac{x\sqrt{3}}{2} + \frac{x-10}{2\pi} = 0 \to 10 = x + x\cdot\pi\sqrt{3} \to x = \frac{10}{1+\pi\sqrt{3}}$$

The length of the piece used to form the triangle is $\dfrac{10}{1+\pi\sqrt{3}}$, the length of the other piece is $\dfrac{10\pi\sqrt{3}}{1+\pi\sqrt{3}}$.

2.

(a) Let $u = 1+\sqrt{y}$, then $du = \dfrac{dy}{2\sqrt{y}}$.

$$\int \frac{dy}{\sqrt{y}\left(1+\sqrt{y}\right)^2} = \int \frac{1}{u^2}\cdot 2\,du = -\frac{2}{u} + c = \boxed{-\frac{2}{1+\sqrt{y}} + c,\ c\in\mathbb{R}}$$

(b) Let $u = \ln(\sin x)$, then $du = \dfrac{1}{\sin x}\cdot\cos x\,dx$.

$$\int_{\pi/6}^{\pi/4} \frac{\cot x}{\ln(\sin x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + c = \Big[\ln|\ln(\sin x)|\Big]_{\pi/6}^{\pi/4}$$

$$= \ln\left|\ln\frac{\sqrt{2}}{2}\right| - \ln\left|\ln\frac{1}{2}\right| = \ln\frac{\left|\ln\frac{\sqrt{2}}{2}\right|}{\left|\ln\frac{1}{2}\right|}$$

$$= \ln\left(\frac{\ln 2 - \ln\sqrt{2}}{\ln 2 - \ln 1}\right) = \ln\frac{1}{2} = \boxed{-\ln 2}$$

(c) Attempt a long polynomial division and split into two integrals.

$$\mathrm{I} = \int \frac{x^3 - 4x - 10}{x^2 - x - 6}\, dx = \int (x+1)\, dx + \int \frac{3x-4}{(x-3)(x+2)}\, dx$$

$$= \frac{x^2}{2} + x + \int \left(\frac{A}{x-3} + \frac{B}{x+2} \right) dx$$

$$\begin{aligned} A(x+2) + B(x-3) &= 3x - 4 \\ x(A+B) + 2A - 3B &= 3x - 4 \end{aligned}$$

$$\left.\begin{aligned} A + B &= 3 \\ 2A - 3B &= -4 \end{aligned}\right\} \quad A = 1, \quad B = 2$$

$$\mathrm{I} = \frac{x^2}{2} + x + \int \left(\frac{1}{x-3} + \frac{2}{x+2} \right) dx = \boxed{\frac{x^2}{2} + x + \ln|x-3| + 2\ln|x+2| + c,\ c \in \mathbb{R}}$$

(d) Let $u = \sec x$, then $du = \tan x \sec x\, dx$.

$$\int \tan x \sec^{123} x\, dx = \int u^{122}\, du = \frac{u^{123}}{123} + c = \boxed{\frac{\sec^{123} x}{123} + c,\ c \in \mathbb{R}}$$

(e) Let $t = \sin^2 x$, then $dt = 2\sin x \cos x\, dx$.

$$\int \sqrt{t - t^2}\, dt = \int \sqrt{\sin^2 x - \sin^4 x} \cdot 2\sin x \cos x\, dx = \int \sqrt{\sin^2 x \cdot (1 - \sin^2 x)} \cdot \sin(2x)\, dx$$

$$= \int \sin x \cos x \cdot \sin(2x)\, dx = \frac{1}{2} \int \sin^2(2x)\, dx = \frac{1}{2} \int \left(1 - \cos^2(2x)\right)\, dx$$

$$= \frac{1}{2} \int \frac{1 - \cos(4x)}{2}\, dx = \frac{1}{4} \left[x - \frac{1}{4}\sin(4x) \right] + c,\ c \in \mathbb{R}$$

We will try to rewrite the result in terms of t.

$$t - t^2 \geq 0 \implies 0 \leq t \leq 1$$

$$t = \sin^2 x \implies \sqrt{t} = \sin x \implies \arcsin\left(\sqrt{t}\right) = x$$

$$t = \sin^2 x = 1 - \cos^2 x \implies \cos x = \sqrt{1-t}$$

$$\sin(4x) = 2\sin(2x)\cos(2x) = 4\sin x \cos x \left(2\cos^2 x - 1\right) = 4\sqrt{t} \cdot \sqrt{1-t} \cdot (1 - 2t)$$

$$\int \sqrt{t-t^2}\,dt = \boxed{\frac{1}{4}\left[\arcsin\left(\sqrt{t}\right) - \sqrt{t(1-t)}\cdot(1-2t)\right] + c,\ c \in \mathbb{R}}$$

(f) Let $u = \sqrt{1+\sqrt{x}}$.

$$u^2 = 1+\sqrt{x} \implies u^2 - 1 = \sqrt{x} \implies \left(u^2-1\right)^2 = x \implies 2\left(u^2-1\right)\cdot 2u\,du = dx$$

$$\int \sqrt{1+\sqrt{x}}\,dx = \int u\cdot(2u^2-2)\cdot 2u\,du = \int \left(4u^4-4u^2\right)\,du = \frac{4u^5}{5} - \frac{4u^3}{3} + c$$

$$= \boxed{\frac{4\left(\sqrt{1+\sqrt{x}}\right)^5}{5} - \frac{4\left(\sqrt{1+\sqrt{x}}\right)^3}{3} + c,\ c\in\mathbb{R}}$$

3.

(a)

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{S\to 1^-}\int_0^S \frac{dx}{\sqrt{1-x}} + \lim_{P\to 1^+}\int_P^2 \frac{dx}{\sqrt{x-1}}$$

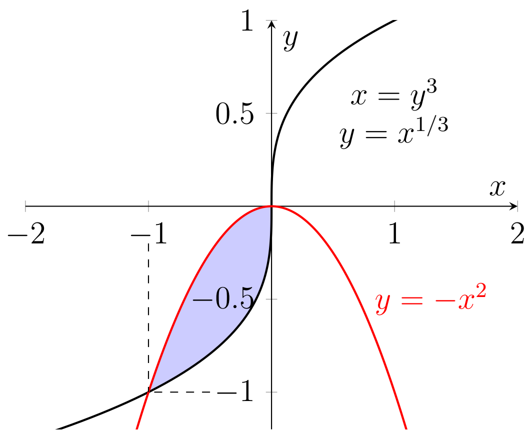
$$= \lim_{S\to 1^-}\left[-2\sqrt{1-x}\right]_0^S + \lim_{P\to 1^+}\left[2\sqrt{x-1}\right]_P^2$$

$$= \lim_{S\to 1^-}\left[-2\sqrt{1-S}\right] + 2\sqrt{1-0} + 2\sqrt{2-1} - \lim_{P\to 1^+}\left[2\sqrt{P-1}\right] = \boxed{4}$$

(b)

$$\int_{-\infty}^{\infty} x^2 \mathrm{e}^{-x^3}\,dx = \lim_{a\to\infty}\int_{-a}^{a} x^2\mathrm{e}^{-x^3}\,dx = \lim_{a\to\infty} -\frac{1}{3}\left[\mathrm{e}^{-x^3}\right]_{-a}^{a} = \lim_{a\to\infty} -\frac{1}{3}\left[\mathrm{e}^{-a^3} - \mathrm{e}^{a^3}\right] = \boxed{\infty}$$

4.



$$V = \int_{-1}^{0} \pi\left[\left(x^{1/3}\right)^2 - \left(-x^2\right)^2\right]\,dx$$

$$= \pi\int_{-1}^{0}\left(x^{2/3} - x^4\right)\,dx$$

$$= \pi\left[\frac{3}{5}x^{5/3} - \frac{x^5}{5}\right]_{-1}^{0}$$

$$= \pi\cdot\left[0 - \left(-\frac{3}{5} + \frac{1}{5}\right)\right] = \boxed{\frac{2\pi}{5}}$$

2015-2016 Fall Midterm (10/12/2015) Solutions
(Last update: 29/08/2025 20:34)

1. Take the first derivative and set to 0.

$$f'(x) = -\frac{1}{x^2} - \frac{2}{x^3}$$
$$f'(x) = 0 \implies \frac{2}{x^3} = -\frac{1}{x^2} \implies x = -2 \quad \text{(candidate for a critical point)}$$

Take the second derivative and set to 0.

$$f''(x) = \frac{2}{x^3} + \frac{6}{x^4}$$
$$f''(x) = 0 \implies \frac{1}{x^3} = -\frac{3}{x^4} \implies x = -3 \quad \text{(candidate for an inflection point)}$$

$\{-2, -3\} \subset \mathcal{D}$. Therefore, $f(-2)$ gives rise to a local extremum. The sign of the first derivative changes from minus to plus, meaning $f(-2)$ is a local minimum. $x = -3$ gives rise to an inflection point because the sign of the second derivative also changes.

Find the asymptotes.

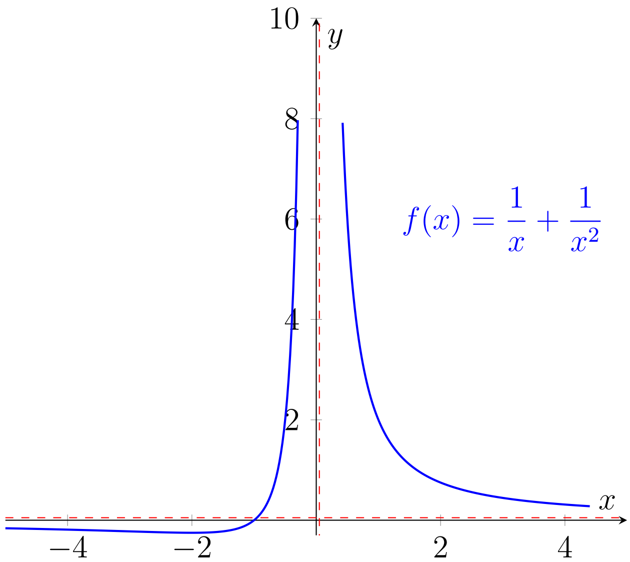
$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$$

$y = 0$ is the horizontal asymptote and $x = 0$ is the vertical asymptote.

Let us find monotonicity and concavity. If the sign of $f'(x)$ is minus, the function is decreasing on the corresponding interval; otherwise, increasing. If the sign of $f''(x)$ is minus, the graph of the function is concave downward; otherwise, concave upward.

x	$(-\infty, -3)$	$(-3, -2)$	$(-2, 0)$	$(0, \infty)$
f' sign	-	-	$+$	-
f'' sign	-	$+$	$+$	$+$

Eventually, sketch the graph.



2. Let $f(x) = 3\tan x + x^3 - 2$. f is continuous on $[0, \pi/4]$ and differentiable on $(0, \pi/4)$. By IVT (Intermediate Value Theorem), there exists at least one point where $f(x) = 0$ because $f(0) = -2$ and $f(\pi/4) = 1 + (\pi/4)^3$. Assume that we have two roots on the interval, so at some point c, $f'(c) = 0$.

$$f'(x) = 3\sec^2 x + 3x^2 \implies f'(c) = 3\sec^2 c + 3c^2 = 0 \implies \sec^2 c = -c^2$$

Since $-c^2 \le 0$ and $\sec^2 c > 0$, there is no c that satisfies the equation. This contradicts our assumption that we have two roots on the interval. By Rolle's theorem, there is only one root on the interval $[0, \pi/4]$.

3. Implicitly differentiate both sides.

$$\frac{d}{dx}\left[x\sin\left(xy - y^2\right)\right] = \frac{d}{dx}\left(x^2 - 1\right)$$

$$1 \cdot \sin\left(xy - y^2\right) + x \cdot \cos\left(xy - y^2\right) \cdot \left[\left(1 \cdot y + x\frac{dy}{dx}\right) - 2y\frac{dy}{dx}\right] = 2x$$

Rearrange the equation to solve for $\frac{dy}{dx}$ through a careful and rigorous attempt.

$$x \cdot \cos\left(xy - y^2\right) \cdot \left[\left(1 \cdot y + x\frac{dy}{dx}\right) - 2y\frac{dy}{dx}\right] = 2x - \sin(xy - y^2)$$

$$y + \frac{dy}{dx}(x - 2y) = \frac{2x - \sin\left(xy - y^2\right)}{x \cdot \cos\left(xy - y^2\right)}$$

$$\frac{dy}{dx}(x - 2y) = \frac{2x - \sin\left(xy - y^2\right)}{x \cdot \cos\left(xy - y^2\right)} - y$$

$$\frac{dy}{dx} = \frac{\frac{2x - \sin(xy - y^2)}{x \cdot \cos(xy - y^2)} - y}{(x - 2y)} \tag{5}$$

Calculate $\left.\frac{dy}{dx}\right|_{(1,1)}$ from (1). This will give us the slope of the tangent line.

$$\left.\frac{dy}{dx}\right|_{(1,1)} = -1$$

Recall: $y - y_0 = m(x - x_0)$. m is $\frac{dy}{dx}$ at $x = 1$. So, the tangent line is:

$$y - 1 = -(x - 1) \implies \boxed{y = 2 - x}$$

4.

(a) Let L be the value of the limit. Then, take the logarithm of both sides.

$$L = \lim_{x \to 0^+} (1 + \sin x)^{\frac{1}{x}}$$

$$\ln(L) = \ln\left[\lim_{x\to 0^+}(1+\sin x)^{\frac{1}{x}}\right]$$

The expression on the right is continuous for $x > 0$. Therefore, we can take the logarithm inside the limit.

$$\ln(L) = \lim_{x\to 0^+}\ln\left[(1+\sin x)^{\frac{1}{x}}\right] = \lim_{x\to 0^+}\left[\frac{\ln(1+\sin x)}{x}\right]$$

If we substitute $x = 0$, the limit is in the form $0/0$. L'Hôpital's rule states that we may take the derivatives of both sides of the fraction if there's a $0/0$ indeterminate form. Apply the chain rule accordingly.

$$\lim_{x\to 0^+}\left[\frac{\ln(1+\sin x)}{x}\right] \overset{\text{L'H.}}{=} \lim_{x\to 0^+}\left[\frac{\frac{1}{1+\sin x}\cdot\cos x}{1}\right] = \lim_{x\to 0^+}\left[\frac{\cos x}{1+\sin x}\right]$$

The limit can now be evaluated by substituting $x = 0$.

$$\lim_{x\to 0^+}\left[\frac{\cos x}{1+\sin x}\right] = \frac{\cos 0}{1+\sin 0} = 1$$

Now, $\ln(L) = 1$. Simply, take L out of the logarithm.

$$\boxed{L = \mathrm{e}}$$

(b) Look at the one-sided limits. Let us first evaluate the limit from the right side. Above and near $x = 3$, the floor function will return 9.

$$\lim_{x\to 3^+}\frac{x^2-\lfloor x^2\rfloor}{x-3} = \lim_{x\to 3^+}\frac{x^2-9}{x-3} = \lim_{x\to 3^+}\frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3^+}(x+3) = 6$$

From the left side, the output is the largest integer less than 9. Therefore, $\lfloor x^2\rfloor = 8$.

$$\lim_{x\to 3^-}\frac{x^2-\lfloor x^2\rfloor}{x-3} = \lim_{x\to 3^-}\frac{x^2-8}{x-3} = -\infty$$

The one-sided limits are not equal to each other. Therefore, the limit does not exist.

5. $x = x(t)$ and $y = y(t)$. The distance between the point P and the origin, and the distance between the point P and the point (7,0) are, respectively, given by:

$$f(t) = \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2}$$

$$g(t) = \sqrt{(x-7)^2+(y-0)^2} = \sqrt{x^2-14x+49+y^2}$$

Take the first derivative with respect to time.

$$f'(t) = \frac{1}{2\sqrt{x^2+y^2}}\cdot\left(2x\frac{dx}{dt}+2y\frac{dy}{dt}\right) \tag{6}$$

$$g'(t) = \frac{1}{2\sqrt{x^2 - 14x + 49 + y^2}} \cdot \left((2x - 14)\frac{dx}{dt} + 2y\frac{dy}{dt} \right) \tag{7}$$

For $t = t_0$, it is given $x(t_0) = 4$, $y(t_0) = 3$ and $f(t_0) = \sqrt{4^2 + 3^2} = 5$, $g(t_0) = \sqrt{3^2 + 3^2} = 3\sqrt{2}$. We then obtain a system of two equations by substituting values in (2) and (3):

$$f'(t_0) = \frac{1}{10} \cdot \left(8\frac{dx}{dt} + 6\frac{dy}{dt} \right) = \sqrt{2}$$

$$g'(t_0) = \frac{1}{6\sqrt{2}} \cdot \left((-6)\frac{dx}{dt} + 6\frac{dy}{dt} \right) = -3$$

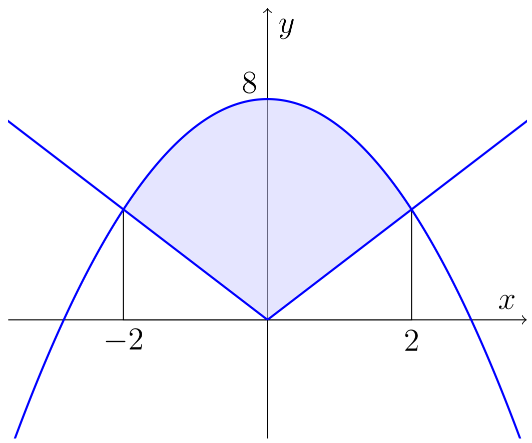
Let us simplify the equations.

$$4x'(t_0) + 3y'(t_0) = 5\sqrt{2}$$
$$-3x'(t_0) + 3y'(t_0) = -9\sqrt{2}$$

The question asks us to find the change in the x-coordinate of P. Therefore, negate the latter equation and solve for $x'(t_0)$.

$$\boxed{x'(t_0) = 2\sqrt{2}}$$

6.



The area can be found by integrating the difference in y with respect to x. We split the integral into two because the absolute value function changes sign.

$$\mathrm{I} = \int_{-2}^{2} (8 - x^2 - 2|x|)\,dx = \int_{-2}^{0} (8 - x^2 + 2x)\,dx + \int_{0}^{2} (8 - x^2 - 2x)\,dx$$

$$\mathrm{I} = \left[8x - \frac{x^3}{3} + x^2 \right]_{-2}^{0} + \left[8x - \frac{x^3}{3} - x^2 \right]_{0}^{2}$$

$$\mathrm{I} = 0 - \left(-16 + \frac{8}{3} + 4 \right) + \left(16 - \frac{8}{3} - 4 \right) - 0 = \boxed{\frac{56}{3}}$$

2016-2017 Fall Midterm (05.12.2016) Solutions
(Last update: 29/08/2025 20:46)

1.

(a) To solve the limit easily, we can use logarithms. The expression is continuous and differentiable for $x > 0$. Assume that the limit exists, then let L be the value of the limit.

$$L = \lim_{x\to\infty} [\mathrm{e}^x + 1]^{\frac{1}{x}}$$

$$\ln(L) = \ln\left(\lim_{x\to\infty} [\mathrm{e}^x + 1]^{\frac{1}{x}}\right) = \lim_{x\to\infty} \left[\ln\left([\mathrm{e}^x + 1]^{\frac{1}{x}}\right)\right]$$

$$\ln(L) = \lim_{x\to\infty} \left[\frac{\ln(\mathrm{e}^x + 1)}{x}\right]$$

The expressions x and $\mathrm{e}^x + 1$ tend to infinity as x approaches infinity. L'Hôpital's rule suggests that we take the derivatives of both sides of the fraction if there's an indeterminate form of infinity ($\frac{\infty}{\infty}$). Using the chain rule, we get:

$$\ln(L) \overset{\text{L'H.}}{=} \lim_{x\to\infty} \left[\frac{\frac{1}{\mathrm{e}^x+1} \cdot \mathrm{e}^x}{1}\right] = \lim_{x\to\infty} \left(\frac{\mathrm{e}^x}{\mathrm{e}^x + 1}\right)$$

It is obvious that $\ln(L) = 1$. e^x grows at the same rate as $\mathrm{e}^x + 1$. The behavior can be confirmed using L'Hôpital's rule. Without further ado, we can convert the logarithm back to its original form.

$$\ln(L) = 1$$

$$\boxed{L = \mathrm{e}}$$

(b) If we substitute for $x = 1$, it is in the form $0/0$. Multiply each side by the conjugate of the denominator.

$$\lim_{x\to 1} \left(\frac{x-1}{\sqrt{x+3}-2} \cdot \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right) = \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{x-1} = \lim_{x\to 1} (\sqrt{x+3}+2) = \boxed{4}$$

2. If f is continuous at a point, then the one-sided limits must be equal to the value of the function at that point. f is constant for $x < 2$, meanwhile polynomial for $x \geq 2$. If we check the point where $x = 2$, the continuity will be provided for all x.

$$\lim_{x\to 2^-} f(x) = 12, \quad \lim_{x\to 2^+} f(x) = a^2x - 2a \implies 12 = 2a^2 - 2a$$

$$2(a^2 - a - 6) = 0 \implies (a-3)(a+2) = 0 \implies \boxed{a_1 = 3,\ a_2 = -2}$$

3. Take logarithms of both sides. Then differentiate implicitly.

$$y(x) = x^{\sin x}$$

$$\ln(y) = \sin x \cdot \ln x$$

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}(\sin x \cdot \ln x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = x^{\sin x}\left(\cos x \cdot \ln x + \sin x \cdot \frac{1}{x}\right) \tag{8}$$

Recall: $y - y_0 = m(x - x_0)$, where m is the slope. We can evaluate (1) at $x = \frac{\pi}{2}$ because $\frac{dy}{dx}\Big|_{(\pi/2,\,\pi/2)}$ gives the rate of change at that point.

$$x_0 = \frac{\pi}{2},\ y_0 = \frac{\pi}{2}$$

$$m = \frac{dy}{dx}\Bigg|_{(\pi/2,\,\pi/2)} = \left(\frac{\pi}{2}\right)^{\sin\frac{\pi}{2}}\left(\cos\frac{\pi}{2} \cdot \ln\frac{\pi}{2} + \sin\frac{\pi}{2} \cdot \frac{2}{\pi}\right) = 1$$

Therefore, the tangent line is $\boxed{y = x}$.

4. Let w, l be the functions of time representing width and length, respectively, Then the area of the rectangle can be written as:

$$A(t) = w(t) \cdot l(t)$$

Take the derivative of both sides with respect to t. Apply the chain rule.

$$\frac{d}{dt}A(t) = \frac{d}{dt}(w(t) \cdot l(t))$$

$$A'(t) = w'(t) \cdot l(t) + w(t) \cdot l'(t)$$

Let t_1 be the moment when as told in the question.

$$l'(t_1) = -3,\ w'(t_1) = 2,\ l(t_1) = 50,\ w(t_1) = 20$$

$$A'(t_1) = 2 \cdot 50 - 20 \cdot 3 = 40$$

$$\boxed{A'(t_1) = 40\,\mathrm{cm}^2/\mathrm{s}}$$

Since $A'(t_1) > 0$, the area increases at $t = t_1$.

5. First off, find the domain. The expression is undefined when the denominator is zero. Therefore, $x^2 - 1 \neq 0 \rightarrow x \neq \pm 1$. Vertical asymptotes occur at $x = \pm 1$.

$$\mathcal{D} = \mathbb{R} - \{-1, 1\}$$

Let us find the limit at infinity.

$$\lim_{x\to\infty} \frac{2x^2}{x^2-1} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{4x}{2x} = \lim_{x\to\infty} 2 = 2$$

Similarly,

$$\lim_{x\to-\infty} \frac{2x^2}{x^2-1} = 2$$

$y = 2$ is the only horizontal asymptote. Now, let us glance at the critical points. This will give us an idea of where the graph becomes stationary. Take the first derivative by applying the quotient rule.

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{2x^2}{x^2-1}\right)$$

$$y' = \frac{4x\cdot(x^2-1) - 2x^2\cdot 2x}{(x^2-1)^2} = -\frac{4x}{(x^2-1)^2}$$

For $x = 0$, $y' = 0$. (0,0) is a critical point. Look for the inflection points now. Take the second derivative.

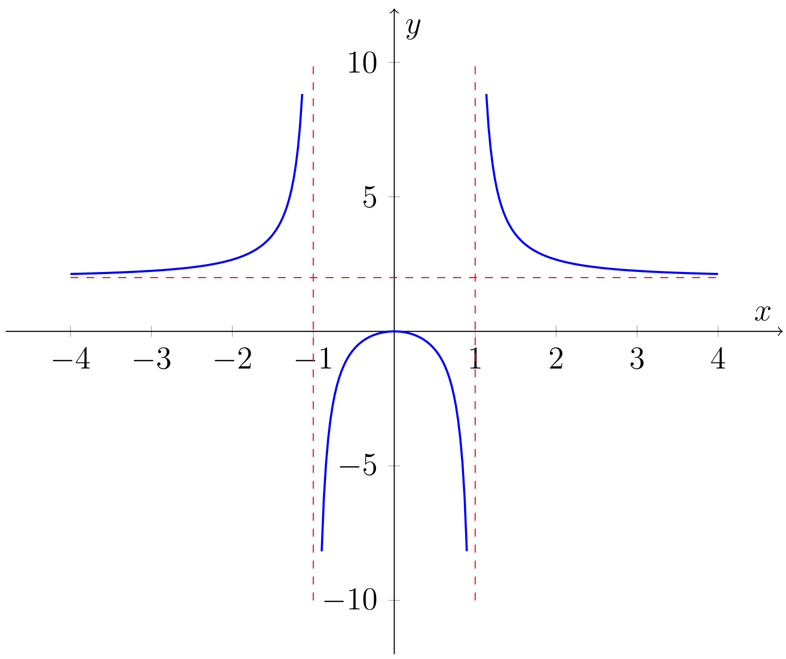
$$y'' = -\frac{4\cdot(x^2-1)^2 - 4x\cdot 2\,(x^2-1)\,(2x)}{(x^2-1)^4} = \frac{12x^2+4}{(x^2-1)^3}$$

$\forall x \in \mathbb{R}$, $12x^2 + 4 > 0$. Therefore, no inflection points.

Eventually, set up a table and see what the graph looks like in certain intervals.

x	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
y	$(2,\infty)$	$(-\infty,0)$	$(-\infty,0)$	$(2,\infty)$
y' sign	+	+	-	-
y'' sign	+	-	-	+

Knowing that $f(0) = 0$, we may sketch the graph.



6. Let $f(x) = x^4+3x+1$. f is continuous everywhere. IVT (Intermediate Value Theorem) states that f takes any value on the interval $[a, b]$ between $f(a)$ and $f(b)$. Let us check $f(-1)$ and $f(-2)$.

$$f(-1) = (-1)^4 + 3(-1) + 1 = -1, \qquad f(-2) = (-2)^4 + 3(-2) + 1 = 11$$

By IVT, there exists at least one point on $[-2, -1]$ such that the value of f is zero there. To show that this is the only point where $f(x) = 0$, we may use MVT (Mean Value Theorem). f is differentiable on $(-2, -1)$ as well as it is continuous on $[-2, -1]$. By MVT, there exists a point c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

where $-2 \leq a \leq -1$ and $-2 \leq b \leq -1$. We can show that $f(x)$ has no more than one root on the interval $(-2, -1)$. Assume that we have a and b such that we have two distinct roots, meaning $f'(c) = 0$ somewhere on $(-2, -1)$. So,

$$f'(x) = 4x^3 + 3 \implies f'(c) = 4c^3 + 3$$

$$f'(c) = 0 \implies 4c^3 + 3 = 0 \implies c = \sqrt[3]{-\frac{3}{4}} > -1$$

The only point where $f(x) = 0$ is $c = \sqrt[3]{-\frac{3}{4}}$; however, it is outside the interval. This contradicts our assumption that we have two distinct roots on $(-2, -1)$. Therefore, there's only one root on the interval $(-2, -1)$.

2017-2018 Summer Midterm (24/07/2018) Solutions
(Last update: 29/08/2025 21:13)

1. Let L be the limit value. Then, take the logarithm of both sides.

$$L = \lim_{x\to 3^+} (x-3)^{\ln(x-2)}$$

$$\ln(L) = \ln\left[\lim_{x\to 3^+} (x-3)^{\ln(x-2)}\right]$$

The function is continuous for $x > 3$. So, we can take the logarithm function inside the limit. Using the property of logarithms, we get:

$$\ln(L) = \lim_{x\to 3^+} \ln\left[(x-3)^{\ln(x-2)}\right] = \lim_{x\to 3^+} \left[\ln(x-2)\cdot\ln(x-3)\right]$$

If we substitute $x = 3$, we see that $\ln(1) = 0$. However, $\ln(x-3)$ becomes undefined (in other words, tends to negative infinity as $x \to 3^+$). We can apply L'Hôpital's rule if we treat these two expressions as a single fraction. Rewrite the limit as follows:

$$\lim_{x\to 3^+} \left[\ln(x-2)\cdot\ln(x-3)\right] = \lim_{x\to 3^+} \left[\frac{\ln(x-2)}{\frac{1}{\ln(x-3)}}\right]$$

The limit is now in the form ∞/∞. Apply the rule where indeterminate forms occur.

$$\lim_{x\to 3^+} \left[\frac{\ln(x-2)}{\frac{1}{\ln(x-3)}}\right] \overset{\text{L'H.}}{=} \lim_{x\to 3^+} \left[\frac{\frac{1}{x-2}}{-\frac{1}{\ln^2(x-3)}\cdot\frac{1}{x-3}}\right] = \lim_{x\to 3^+} \left[-\frac{\ln^2(x-3)}{\frac{x-2}{x-3}}\right] \quad \left[\frac{\infty}{\infty}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 3^+} \left[-\frac{2\ln(x-3)\cdot\frac{1}{x-3}}{\frac{(x-3)-(x-2)}{(x-3)^2}}\right] = \lim_{x\to 3^+} \left[\frac{2\ln(x-3)}{\frac{1}{x-3}}\right] \quad \left[\frac{\infty}{\infty}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 3^+} \left[\frac{\frac{2}{x-3}}{-\frac{1}{(x-3)^2}}\right] = \lim_{x\to 3^+} \left[-\frac{2(x-3)}{1}\right] = \lim_{x\to 3^+} (6-2x)$$

$$= 0$$

Recall that we evaluated $\ln(L) = 0$, so $\boxed{L = 1}$.

2. We need to ensure continuity at $x = 1$. The one-sided limits must be equal to the value of the function at that point. The function is a polynomial expression for $x < 1$, and another polynomial expression for $x > 1$. Both expressions are defined for $x = 1$ (x=1 is actually in the domain with another condition), so we can just substitute $x = 1$ in the limits.

$$\lim_{x\to 1^+} (ax + b) = a + b = 3$$

$$\lim_{x\to 1^-}(x^2-4x+b+3) = 1^2-4+b+3 = 3$$

$\boxed{b=3}$, so $a+3=3 \to \boxed{a=0}$.

Remark: Condition $f(2)+3=f(0)$ is redundant because we already found the values of a and b. It does not provide any further useful information, and it is unknown why it was included in the question.

3. Let $x=f(t)$, $y=g(t)$ and L be the length of the ladder. $f(t)$ and $g(t)$ represent the location of the bottom and top of the ladder, respectively. The length of the ladder remains constant as time goes on. We can write the following using the Pythagorean theorem. Apply the Chain Rule appropriately.

$$L=\sqrt{f^2(t)+g^2(t)}$$

$$\frac{dL}{dt}=\frac{d}{dt}\sqrt{f^2(t)+g^2(t)}$$

$$0=\frac{1}{2\sqrt{f^2(t)+g^2(t)}}\cdot[2f(t)f'(t)+2g(t)g'(t)]$$

$$\therefore\ f(t)f'(t)=-g(t)g'(t) \tag{9}$$

For $t=t_0$, given $g(t_0)=9$ m, $f'(t_0)=-6$ m/s, $g'(t_0)=5$ m/s. Find $f(t_0)$ using (1).

$$f(t_0)=-\frac{g(t_0)g'(t_0)}{f'(t_0)}=-\frac{9\cdot(-5)}{6}=\frac{15}{2}$$

Now, we can find L.

$$L=\sqrt{f^2(t_0)+g^2(t_0)}=\sqrt{\left(\frac{15}{2}\right)^2+9^2}=\sqrt{\frac{549}{4}}$$

$$\boxed{L=\frac{3\sqrt{61}}{2}}$$

4. First off, find the domain. The expression is undefined when the denominator is zero. Therefore, $x^2\neq 0 \to x\neq 0$. The only vertical asymptote occurs at $x=0$.

$$\mathcal{D}=\mathbb{R}-\{0\}$$

Let us find the limit at infinity.

$$\lim_{x\to\infty}\frac{x^2+1}{x}\overset{\text{L'H.}}{=}\lim_{x\to\infty}\frac{2x}{1}=\infty$$

Similarly,

$$\lim_{x\to-\infty}\frac{x^2+1}{x}=-\infty$$

There is no horizontal asymptote. However, there exists a slant asymptote. If we attempt to make a long division, the quotient will be x. So, the slant asymptote is $y = x$. Let us verify with the limit.

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x} - x\right) = \lim_{x\to\infty}\frac{1}{x} = 0$$

Take the first derivative by applying the quotient rule.

$$y' = \frac{2x\cdot x - (x^2+1)}{x^2} = \frac{x^2-1}{x^2}$$

y' is undefined for $x = 0$, and $y' = 0$ for $x = \pm 1$. Since 0 is not in the domain, the *only* critical points are $x = \pm 1$.

Take the second derivative.

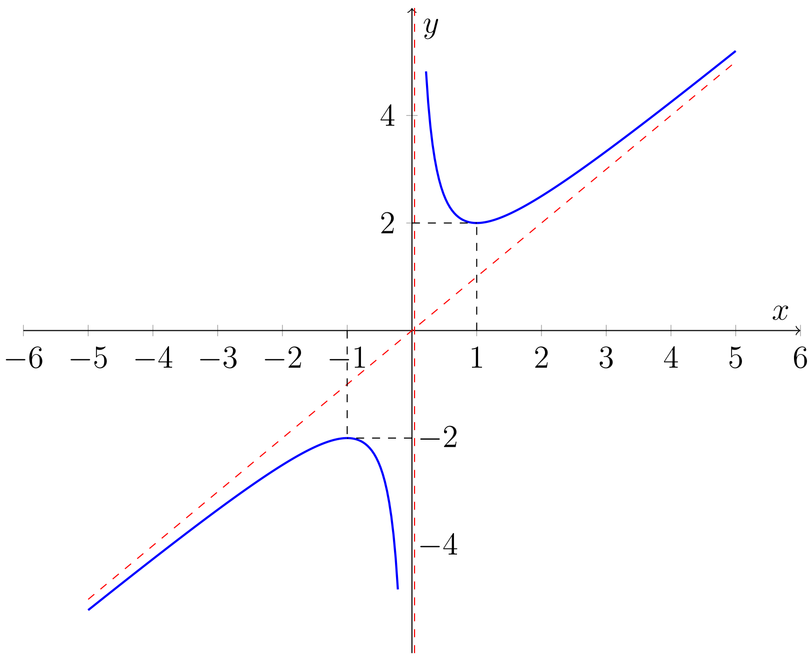
$$y'' = \frac{2x\cdot x^2 - (x^2-1)\cdot(2x)}{x^4} = \frac{1}{x^3}$$

There is no inflection point because $\dfrac{1}{x^3} \neq 0,\ \forall x \in \mathbb{R}$.

Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f(-1) = \frac{(-1)^2+1}{-1} = -2, \quad f(1) = \frac{(1)^2+1}{1} = 2$$

x	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
y	$(-\infty,-2)$	$(-\infty,-2)$	$(2,\infty)$	$(2,\infty)$
y' sign	+	-	-	+
y'' sign	-	-	+	+



5.

(a) Let $u = \sqrt{4-\sqrt{x}}$.

$$u^2 = 4-\sqrt{x} \implies 4-u^2 = \sqrt{x} \implies \left(4-u^2\right)^2 = x \implies 2\left(4-u^2\right)\cdot(-2u)\,du = dx$$

$$\int \sqrt{4-\sqrt{x}}\,dx = \int u\cdot\left(8-2u^2\right)\cdot(-2u)\,du = \int \left(4u^4-16u^2\right)\,du = \frac{4u^5}{5} - \frac{16u^3}{3} + c$$

$$= \boxed{\frac{4\left(\sqrt{4-\sqrt{x}}\right)^5}{5} - \frac{16\left(\sqrt{4-\sqrt{x}}\right)^3}{3} + c, \quad c\in\mathbb{R}}$$

(b) Apply integration by parts.

$$\left.\begin{array}{c} u = (\ln x)^2 \implies du = 2\ln x\cdot\dfrac{1}{x}\,dx \\ dv = dx \implies v = x \end{array}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2\ln x\,dx$$

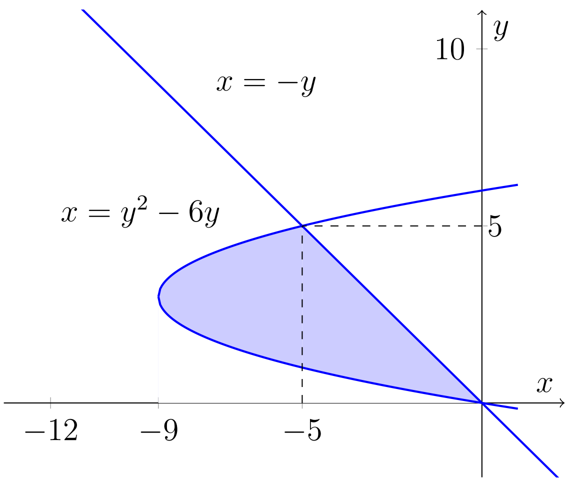
Apply integration by parts once again.

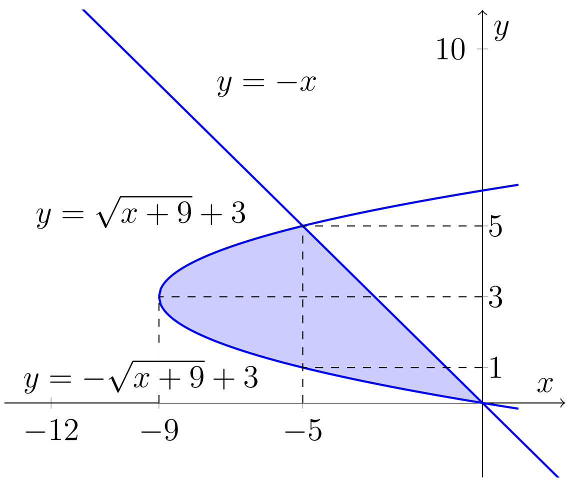
$$\left.\begin{array}{c} u = \ln x \implies du = \dfrac{1}{x}\,dx \\ dv = dx \implies v = x \end{array}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2\ln x\,dx$$

$$x(\ln x)^2 - \int 2\ln x\,dx = x(\ln x)^2 - 2\left[x\ln x - \int dx\right] = \boxed{x(\ln x)^2 - 2x\ln x + 2x + c,\ c\in\mathbb{R}}$$

6.





(i) We'll take the integral along the y-axis. The lower and upper limits of the integral are 0 and 5, respectively.

$$A = \int_0^5 [(-y) - (y^2 - 6y)]\, dy = \int_0^5 (-y^2 + 5y)\, dy$$

(ii) We have two different regions. $y = -x$ and $y = \sqrt{x-9} + 3$ intersect at the point $(-5, 5)$. Therefore, we will write two separate integrals.

$$A = \int_{-9}^{-5} \left[\left(\sqrt{x+9} + 3\right) - \left(-\sqrt{x+9} + 3\right)\right]\, dx + \int_{-5}^{0} \left[(-x) - \left(-\sqrt{x+9} + 3\right)\right]\, dx$$

$$= \int_{-9}^{-5} 2\left(\sqrt{x+9}\right)\, dx + \int_{-5}^{0} \left(-x + \sqrt{x+9} - 3\right)\, dx$$

2019-2020 Fall Midterm (04/11/2019) Solutions
(Last update: 29/08/2025 21:51)

1. Let L be the value of the limit.

$$L = \lim_{x\to 3^+} \cos(x-3)^{\ln\left(\frac{2x}{3}-2\right)}$$
$$\ln(L) = \ln\left(\lim_{x\to 3^+} \cos(x-3)^{\ln\left(\frac{2x}{3}-2\right)}\right)$$

Since $\cos(x-3)^{\ln(2x/3-2)}$ is continuous for $x > 3$, we can take the logarithm function inside the limit. Using the property of logarithms, we get:

$$\ln(L) = \lim_{x\to 3^+}\left[\ln\left(\cos(x-3)^{\ln\left(\frac{2x}{3}-2\right)}\right)\right] = \lim_{x\to 3^+}\left[\ln\left(\cos(x-3)\right)\ln\left(\frac{2x}{3}-2\right)\right] \quad [0\cdot\infty]$$

Rearrange the limit to obtain an indeterminate form. Afterwards, apply L'Hôpital's rule.

$$\lim_{x\to 3^+}\left[\ln\left(\cos(x-3)\right)\ln\left(\frac{2x}{3}-2\right)\right] = \lim_{x\to 3^+}\frac{\ln\left(\frac{2x}{3}-2\right)}{1/\ln\left(\cos(x-3)\right)} \quad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 3^+}\frac{\dfrac{1}{\frac{2x}{3}-2}\cdot\dfrac{2}{3}}{\left(-\ln^{-2}\cos(x-3)\right)\cdot\dfrac{1}{\cos(x-3)}\cdot\left(-\sin(x-3)\right)}$$

$$= \lim_{x\to 3^+}\frac{\ln^2(\cos(x-3))}{(x-3)\tan(x-3)} \quad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 3^+}\frac{2\ln(\cos(x-3))\cdot\dfrac{1}{\cos(x-3)}\cdot(-\sin(x-3))}{\tan(x-3)+(x-3)\sec^2(x-3)}$$

$$= \lim_{x\to 3^+}\frac{-2\ln(\cos(x-3))\sin(x-3)}{\sin(x-3)+(x-3)\sec(x-3)} \overset{u=x-3}{=} \lim_{u\to 0^+}\left(\frac{-2\ln(\cos(u))\sin(u)}{\sin(u)+u\sec(u)}\cdot\frac{u}{u}\right)$$

$$= \frac{\lim\limits_{u\to 0^+}\left[-2\ln(\cos(u))\right]\cdot\lim\limits_{u\to 0^+}\dfrac{\sin(u)}{u}}{\lim\limits_{u\to 0^+}\sec(u)+\lim\limits_{u\to 0^+}\dfrac{\sin(u)}{u}} \quad \left[\lim_{u\to 0}\frac{\sin(u)}{u}=1\right]$$

$$= \frac{\lim\limits_{u\to 0^+}\left[-2\ln(\cos(u))\right]}{\lim\limits_{u\to 0^+}\sec(u)+1} = \frac{-2\ln(\cos(0))}{1+1} = \frac{-2\ln(1)}{2} = 0$$

We found out that $\ln(L) = 0$. Therefore, $\boxed{L=1}$.

2. To ensure continuity at $x = 0$, the one-sided limit values must be equal to the value of the function at that point.

$$\lim_{x\to 0^-} \frac{\tan ax}{\tan bx} = \lim_{x\to 0^+} (ax + b) = f(0) = 4$$

The easy part is that we can calculate the limit from the right.

$$\lim_{x\to 0^+} (ax + b) = 0 + b = b$$

Hence, $b = 4$. To calculate from the left, we need another technique.

$$\lim_{x\to 0^-} \frac{\tan ax}{\tan bx} = \lim_{x\to 0^-} \left(\frac{\sin ax}{\cos ax} \cdot \frac{\cos bx}{\sin bx} \cdot \frac{bx}{bx} \cdot \frac{ax}{ax} \right)$$

$$= \lim_{x\to 0^-} \left(\frac{\sin ax}{ax} \right) \cdot \lim_{x\to 0^-} \left(\frac{1}{\frac{\sin bx}{bx}} \right) \cdot \lim_{x\to 0^-} \left(\frac{\cos(bx) \cdot ax}{\cos(ax) \cdot bx} \right)$$

$$= 1 \cdot \frac{1}{\lim_{x\to 0^-} \dfrac{\sin bx}{bx}} \cdot \lim_{x\to 0^-} \left(\frac{\cos(bx) \cdot a}{\cos(ax) \cdot b} \right) = 1 \cdot 1 \cdot \left(\frac{\cos(0) \cdot a}{\cos(0) \cdot b} \right)$$

$$= \frac{a}{b}$$

Now, set $\frac{a}{b} = b \rightarrow a = 16$. $\boxed{a = 16,\ b = 4}$

3. $-1 \leq \sin x \leq 1$, and $\sin x$ is continuous $\forall x \in \mathbb{R}$. $1 - 2x$ is continuous everywhere and takes any value in $\mathbb{R}$. Therefore, the equation $\sin x = 1 - 2x$ must have at least one real solution by IVT, and the y-intercept is on the interval $[-1, 1]$.

Let $f(x) = \sin x - 1 + 2x$ and x_1 be one solution to the equation. Then, the root must satisfy $|1 - 2x| \leq 1$. To disprove the existence of another root, we assume that x_2 is another distinct root. Since $f(x_1) = f(x_2) = 0$ and $f(x)$ is differentiable everywhere, by Rolle's theorem, there must exist a point c such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

Take the first derivative and calculate $f'(c)$

$$f'(c) = \cos c + 2$$

$-1 \leq \cos x \leq 1$. Therefore, there is no such c that satisfies $f'(c) = 0$. This is a contradiction. By Rolle's theorem, there is *only* one root satisfying $\sin x = 1 - 2x$.

4. Let $f(t)$ and $g(t)$ represent the distance between Ship A and point O, and the distance between Ship B and point O, respectively. The distance between the ships can be represented using the Pythagorean theorem as follows:

$$D^2(t) = f^2(t) + g^2(t)$$

Take the derivative of both sides.

$$2D\frac{dD}{dt} = 2f(t)f'(t) + 2g(t)g'(t)$$

Solve for $\frac{dD}{dt}$.

$$\frac{dD}{dt} = \frac{f(t)f'(t) + g(t)g'(t)}{D}$$

For $t = t_0$, we have $f(t_0) = 60$, $g(t_0) = 80, f'(t_0) = 20$, $g'(t_0) = -25$, $D(t_0) = \sqrt{60^2 + 80^2} = 100$. We may now find the rate of change of the distance at that time.

$$\left.\frac{dD}{dt}\right|_{t=t_0} = \frac{60 \cdot 20 - 80 \cdot 25}{100} = \boxed{-8\,\text{miles/hour}}$$

5. First off, find the domain. The expression is undefined when the denominator is zero. Therefore, $x \neq 0$. The only vertical asymptote occurs at $x = 0$.

$$\mathcal{D} = \mathbb{R} - \{0\}$$

Let us find the limit at infinity and the limit at negative infinity.

$$\lim_{x\to\infty} \frac{\mathrm{e}^x}{x} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{\mathrm{e}^x}{1} = \infty$$

$$\lim_{x\to-\infty} \frac{\mathrm{e}^x}{x} = 0$$

The horizontal asymptote occurs only at $y = 0$.

Take the first derivative by applying the quotient rule.

$$y' = \frac{\mathrm{e}^x \cdot x - \mathrm{e}^x \cdot 1}{x^2} = \frac{\mathrm{e}^x(x-1)}{x^2}$$

y' is undefined for $x = 0$, and $y' = 0$ for $x = 1$. Since 0 is not in the domain, the *only* critical point is $x = 1$.

Take the second derivative.

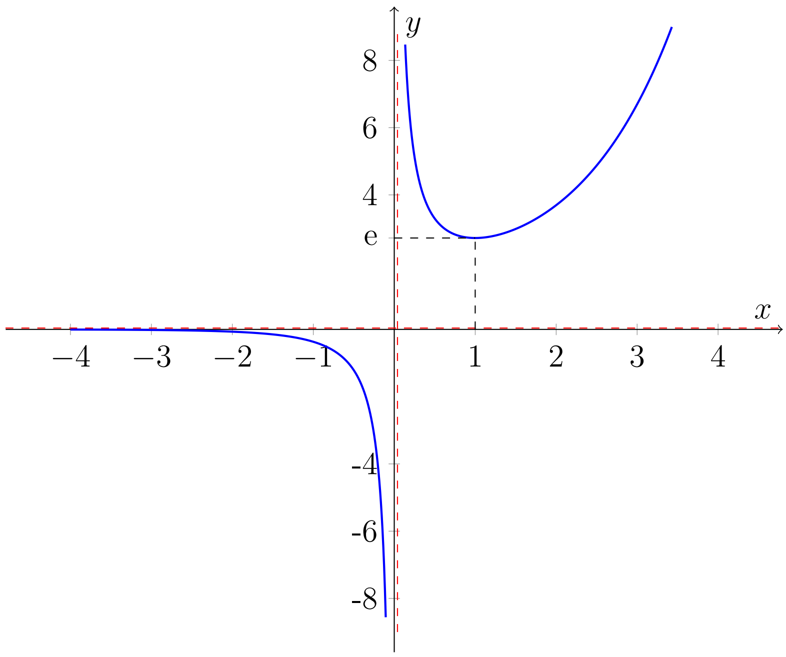
$$y'' = \frac{[\mathrm{e}^x(x-1) + \mathrm{e}^x]x^2 - \mathrm{e}^x(x-1) \cdot 2x}{x^4} = \frac{\mathrm{e}^x(x^2 - 2x + 2)}{x^3}$$

There is no inflection point.

Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f(1) = \frac{\mathrm{e}^1}{1} = \mathrm{e}$$

x	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
y	$(-\infty, 0)$	(∞, e)	(e, ∞)
y' sign	-	-	+
y'' sign	-	+	+



6.

(a) Let $x = 3\tan u$, then $dx = 3\sec^2 u\, du$.

$$\mathrm{I} = \int x^2\sqrt{9+x^2}\,dx = \int (3\tan u)^2 \cdot \sqrt{9+(3\tan u)^2} \cdot 3\sec^2(u)\,du \quad [1+\tan^2 u = \sec^2 u]$$

$$= 81\int \tan^2 u \cdot \sec^3 u\,du = 81\int \frac{\sin^2 u}{\cos^5 u}\,du = 81\int \frac{1-\cos^2 u}{\cos^5 u}\,du$$

$$= 81\int \sec^5 u\,du - 81\int \sec^3 u\,du \tag{10}$$

Find the left-hand integral in (1) with integration by parts.

$$w = \sec^3 u \;\rightarrow\; dw = 3\sec^3 u \tan u\,du$$
$$dz = \sec^2 u\,du \;\rightarrow\; z = \tan u$$

$$\int \sec^5 u\,du = \tan u \cdot \sec^3 u - 3\int \tan^2 u \cdot \sec^3 u\,du = \tan u \cdot \sec^3 u - 3\int (\sec^5 u - \sec^3 u)\,du$$

The integral we want to evaluate appears on the right side. After a little algebra, we get:

$$\int \sec^5 u\, du = \frac{1}{4} \cdot \tan u \cdot \sec^3 u + \frac{3}{4} \int \sec^3 u\, du$$

Rewrite (1) and calculate the other integral in (1) with integration by parts.

$$\mathrm{I} = \frac{81}{4} \cdot \tan u \cdot \sec^3 u - \frac{81}{4} \int \sec^3 u\, du \tag{11}$$

$$\begin{aligned} w = \sec u \;&\rightarrow\; dw = \sec u \tan u\, du \\ dz = \sec^2 u\, du \;&\rightarrow\; z = \tan u \end{aligned}$$

$$\int \sec^3 u\, du = \tan u \cdot \sec u - \int \tan^2 u \sec u\, du = \tan u \cdot \sec u - \int \frac{1 - \cos^2 u}{\cos^3 u}\, du$$

$$= \tan u \cdot \sec u - \int \sec^3 u\, du + \int \sec u\, du$$

We encountered a similar case when calculating $\int \sec^5 u\, du$. So,

$$\int \sec^3 u\, du = \frac{1}{2} \cdot \tan u \cdot \sec u + \frac{1}{2} \cdot \int \sec u\, du$$

The integral of $\sec u$ with respect to u is as follows. One can derive it with particular methods.

$$\int \sec u\, du = \ln|\tan u + \sec u| + c_1,\ c_1 \in \mathbb{R} \tag{12}$$

Rewrite (2) using (3).

$$\mathrm{I} = \frac{81}{4} \cdot \tan u \cdot \sec^3 u - \frac{81}{8} \cdot \tan u \cdot \sec u - \frac{81}{8} \cdot \ln|\tan u + \sec u| + c$$

Recall that $x = 3\tan u$, then $x^2 = 9\tan^2 u = 9\sec^2 u - 9 \;\rightarrow\; \sec u = \dfrac{\sqrt{x^2+9}}{3}$. The result is then as follows. Furthermore, we can omit the constant part to simplify.

$$\mathrm{I} = \frac{x\sqrt{x^2+9}}{8}\left(2x^2+9\right) - \frac{81}{8}\ln\left|\frac{x+\sqrt{x^2+9}}{3}\right| + c,\ c \in \mathbb{R}$$

$$\boxed{\mathrm{I} = \frac{x\sqrt{x^2+9}}{8}\left(2x^2+9\right) - \frac{81}{8}\ln\left|x+\sqrt{x^2+9}\right| + c,\ c \in \mathbb{R}}$$

(b) Rewrite the expression. Then, let $u = \tan^2 x + 1$. So, $du = 2\tan x \sec^2 x\, dx$

$$\mathrm{I} = \int \tan x \cdot \sec^6 x\, dx \qquad \left[\tan^2 x + 1 = \sec^2 x\right]$$

$$= \int \tan x \cdot \sec^2 x \cdot (1 + \tan^2 x)^2\, dx = \frac{1}{2}\int u^2\, du = \frac{u^3}{6} + c$$

$$\boxed{\mathrm{I} = \frac{(\tan^2 x + 1)^3}{6} + c,\ c \in \mathbb{R}}$$

(c) This is an improper integral; we need to make use of the limit concept. The expression is undefined for $x = 4$.

$$\mathrm{I} = \int_4^8 \frac{1}{(x-4)^3}\, dx = \lim_{\mathrm{R}\to 4^+} \int_{\mathrm{R}}^8 \frac{1}{(x-4)^3}\, dx = \lim_{\mathrm{R}\to 4^+} \left[-\frac{1}{2(x-4)^2}\right]_{\mathrm{R}}^8$$

$$= \lim_{\mathrm{R}\to 4^+} \left[-\frac{1}{32} + \frac{1}{(R-4)^2}\right] = \boxed{\infty}$$

(d) We'll use integration by parts.

$$u = e^x \ \to\ du = \mathrm{e}^x\, dx$$
$$dv = \mathrm{e}^x \sin \mathrm{e}^x\, dx \ \to\ v = -\cos \mathrm{e}^x$$

$$\int \mathrm{e}^{2x} \sin \mathrm{e}^x\, dx = (-\cos \mathrm{e}^x) \cdot \mathrm{e}^x - \int \mathrm{e}^x \cdot (-\cos \mathrm{e}^x)\, dx$$

$$= \boxed{-\mathrm{e}^x \cos \mathrm{e}^x + \sin \mathrm{e}^x + c,\ c \in \mathbb{R}}$$

(e) We may utilize the tangent half-angle substitution, which is sometimes called the Weierstrass substitution. Let $t = \tan\left(\dfrac{x}{2}\right)$. After some mathematical operations, we get the following. One can later derive the formulas.

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}\, dt$$

Rewrite the integral and apply partial fraction decomposition.

$$\mathrm{I} = \int \frac{dx}{\sin x - \cos x} = \int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}\, dt = \int \frac{2}{t^2 + 2t - 1}\, dt = 2\int \frac{1}{(t+1)^2 - (\sqrt{2})^2}\, dt$$

$$= 2\int \frac{1}{\left(t+1-\sqrt{2}\right)\left(t+1+\sqrt{2}\right)}\, dt = 2\int \left(\frac{A}{t+1+\sqrt{2}} + \frac{B}{t+1-\sqrt{2}}\right) dt \qquad (13)$$

$$A(t+1-\sqrt{2}) + B(t+1+\sqrt{2}) = 1$$
$$t(A+B) + A + B + \sqrt{2}(B-A) = 1$$
$$\therefore A+B=0 \quad [\text{eliminate } t] \;\rightarrow\; B-A = \frac{1}{\sqrt{2}}$$

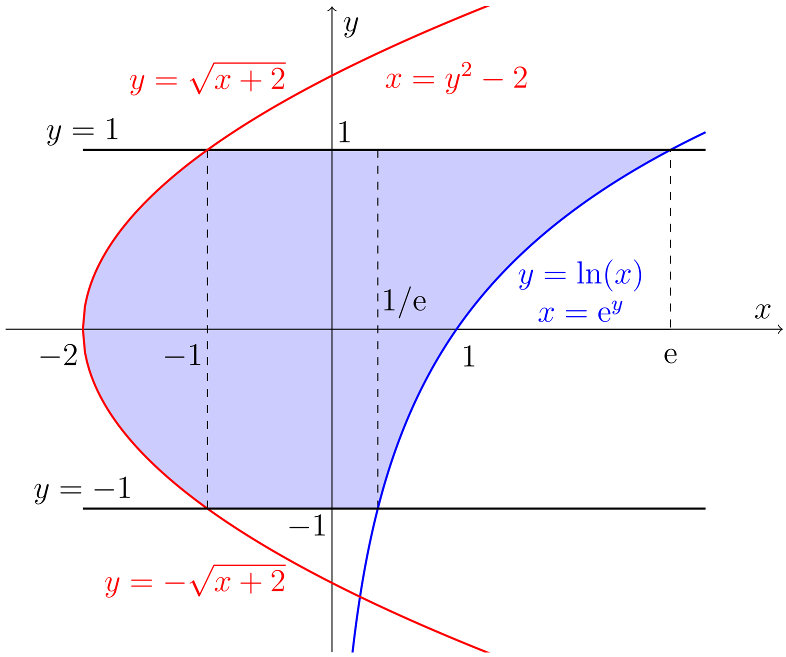
$$\left.\begin{aligned} A+B &= 0 \\ B-A &= \frac{1}{\sqrt{2}} \end{aligned}\right\} \quad A = -\frac{1}{2\sqrt{2}}, \quad B = \frac{1}{2\sqrt{2}}$$

Plug the values of A and B into (4).

$$\mathrm{I} = \frac{\sqrt{2}}{2}\int\left(\frac{1}{t+1-\sqrt{2}} - \frac{1}{t+1+\sqrt{2}}\right) dt = \frac{\sqrt{2}}{2}\ln\left(\frac{|t+1-\sqrt{2}|}{|t+1+\sqrt{2}|}\right) + c,\ c\in\mathbb{R}$$

$$\boxed{\mathrm{I} = \frac{\sqrt{2}}{2}\ln\left(\frac{\left|\tan\left(\frac{x}{2}\right)+1-\sqrt{2}\right|}{\left|\tan\left(\frac{x}{2}\right)+1+\sqrt{2}\right|}\right) + c, \quad c\in\mathbb{R}}$$

7.



(i) The variable is y. Hence, the limits are $-1, 1$, respectively.

$$A = \int_{-1}^{1}\left[\mathrm{e}^y - (y^2-2)\right] dy$$

(ii) We have three different regions. This leads us to take three different integrals.

$$A = \int_{-2}^{-1}\left[\sqrt{x+2} - (-\sqrt{x+2})\right] dx + \int_{-1}^{1/\mathrm{e}}[1-(-1)]\,dx + \int_{1/\mathrm{e}}^{\mathrm{e}}(1-\ln x)\,dx$$

2020-2021 Fall Midterm (30/11/2020) Solutions
(Last update: 29/08/2025 22:18)

1. Let L be the value of the limit. Since the expression is continuous for $x > 0$, we can apply the logarithm function to each side of the equation. Then, we can swap the logarithm and the limit. Use the property of logarithms afterwards.

$$L = \lim_{x \to 0^+} \left(\sqrt{x}\right)^{\ln(x+1)}$$

$$\ln(L) = \ln\left[\lim_{x \to 0^+} \left(\sqrt{x}\right)^{\ln(x+1)}\right] = \lim_{x \to 0^+} \ln\left[\left(\sqrt{x}\right)^{\ln(x+1)}\right] = \lim_{x \to 0^+} \ln\left[\left(\sqrt{x}\right)^{\ln(x+1)}\right]$$

$$\ln(L) = \lim_{x \to 0^+} \left[\ln(x+1) \cdot \ln\left(\sqrt{x}\right)\right] \quad [0 \cdot \infty]$$

Make it so the limit is in the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ in order to apply the L'Hôpital's rule.

$$\ln(L) = \lim_{x \to 0^+} \left[\frac{\ln\left(\sqrt{x}\right)}{\frac{1}{\ln(x+1)}}\right] \quad \left[\frac{\infty}{\infty}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x \to 0^+} \left[\frac{\frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}}{-\frac{1}{\ln^2(x+1)} \cdot \frac{1}{x+1}}\right] = \lim_{x \to 0^+} \left[-\frac{\ln^2(x+1) \cdot (x+1)}{2x}\right]$$

$$= \lim_{x \to 0^+} \left[-\frac{\ln^2(x+1)}{2x}\right] \cdot \lim_{x \to 0^+} (x+1) = \lim_{x \to 0^+} \left[-\frac{\ln^2(x+1)}{2x}\right] \quad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x \to 0^+} \left[-\frac{2\ln(x+1) \cdot \frac{1}{x+1}}{2}\right] = \lim_{x \to 0^+} \left[\frac{\ln(x+1)}{x+1}\right] = \frac{\ln 1}{1} = 0$$

$\ln(L) = 0$, so $\boxed{L = 1}$.

2. $\arctan \dfrac{1}{x}$ takes it values on $-\dfrac{\pi}{2} \leq \arctan \dfrac{1}{x} \leq \dfrac{\pi}{2}$. Multiply each side by x, then we get $-\dfrac{x\pi}{2} \leq x \arctan \dfrac{1}{x} \leq \dfrac{x\pi}{2}$. Take the limits of each side. By the squeeze theorem, we see that the limit of $x \arctan \dfrac{1}{x}$ at the point $x = 0$ is 0. This means that for $f(x)$, the limit from the right side also equals 0.

$$\lim_{x \to 0} \left(-\frac{x\pi}{2}\right) \leq \lim_{x \to 0} \left(x \arctan \frac{1}{x}\right) \leq \lim_{x \to 0} \left(\frac{x\pi}{2}\right)$$

$$0 \leq \lim_{x \to 0} \left(x \arctan \frac{1}{x}\right) \leq 0$$

$$\therefore \lim_{x \to 0} \left(x \arctan \frac{1}{x}\right) = 0$$

From the left side, the limit is equal to as follows.

$$\lim_{x\to 0^-}\frac{x-\cos x}{x^2}=\lim_{x\to 0^-}(x-\cos x)\cdot\lim_{x\to 0^-}\frac{1}{x^2}=-\infty$$

Continuity requires the equality of one-sided limits and the value of the function at that point. However, the one-sided limits are not equal; $0\neq-\infty$. Therefore, $f(x)$ is discontinuous at $x=0$.

3. Differentiate both sides implicitly.

$$\begin{aligned}\frac{d}{dx}\left(\cos y^2+xy+1\right)&=\frac{d}{dx}(0)\\ -\sin y^2\cdot 2y\frac{dy}{dx}+y+x\frac{dy}{dx}&=0\\ \frac{dy}{dx}\left(-\sin y^2\cdot 2y+x\right)&=-y\\ \frac{dy}{dx}&=\frac{y}{\sin y^2\cdot 2y-x}\end{aligned}\tag{14}$$

Evaluate $\dfrac{dy}{dx}$ at the point.

$$\left.\frac{dy}{dx}\right|_{\left(\sqrt{\frac{2}{\pi}},-\sqrt{\frac{\pi}{2}}\right)}=\frac{y}{\sin y^2\cdot 2y-x}=\frac{-\sqrt{\frac{\pi}{2}}}{\sin\left(\left(-\sqrt{\frac{\pi}{2}}\right)^2\right)\cdot 2\left(-\sqrt{\frac{\pi}{2}}\right)-\sqrt{\frac{2}{\pi}}}=\frac{\sqrt{\frac{\pi}{2}}}{\sqrt{2\pi}+\sqrt{\frac{2}{\pi}}}\tag{15}$$

Recall: $y-y_0=m(x-x_0)$, where m is the slope. Substitute m with (2) and find the tangent line.

$$\boxed{y+\sqrt{\frac{\pi}{2}}=\frac{\sqrt{\frac{\pi}{2}}}{\sqrt{2\pi}+\sqrt{\frac{2}{\pi}}}\left(x-\sqrt{\frac{2}{\pi}}\right)}$$

4. Let $S(t)$, $V(t)$, $a(t)$ represent the surface area, the volume, and the length of one side of the object, respectively, as a function of time. We may write the following.

$$S(t)=6a^2(t),\quad V(t)=a^3(t)$$

Given that at $t=t_0$, $V(t_0)=216$, $S'(t_0)=-36$. Using the relationship with the sides,

$$\begin{aligned}&V(t_0)=a^3(t_0)=216\ \to\ a(t_0)=6\\ &S'(t_0)=12a(t_0)a'(t_0)=-36\\ &\therefore 12\cdot 6\cdot a'(t_0)=-36\ \to\ a'(t_0)=-\frac{1}{2}\end{aligned}$$

$$\boxed{a'(t_0)=-\frac{1}{2}\,\mathrm{cm/h}}$$

5.

(a) Let $f(x) = \mathrm{e}^x + x$. f is continuous and differentiable for all $x \in \mathbb{R}$.

$$f(-1) = \mathrm{e}^{-1} - 1 = \frac{1}{\mathrm{e}} - 1, \quad f(0) = \mathrm{e} - 0 = \mathrm{e}$$

Since $f(-1) < 0$ and $f(0) > 0$ and f is continuous on the interval $[-1, 0]$, by IVT, there is at least one point x_1 that satisfies $f(x_1) = 0$. Assume that there is another distinct root x_2. Rolle's theorem states that if f is continuous on a particular interval with endpoints having the same function value, there exists a point c on that interval such that $f'(c) = 0$ there.

$$f'(c) = \mathrm{e}^c + 1 \geq 1 \qquad [\mathrm{e}^c > 0]$$

This yields a contradiction. Therefore, there is *only* one root.

(b) The expression is defined $\forall x \in \mathbb{R}$. Let us find the limit at infinity and the limit at negative infinity.

$$\lim_{x\to\infty} (\mathrm{e}^x + x) = \infty \qquad \lim_{x\to-\infty} (\mathrm{e}^x + x) = -\infty$$

There are no vertical or horizontal asymptotes. However, there is a slant asymptote. Attempt a long polynomial division and we will find that the slant asymptote is $y = x$. Verify with the following limit:

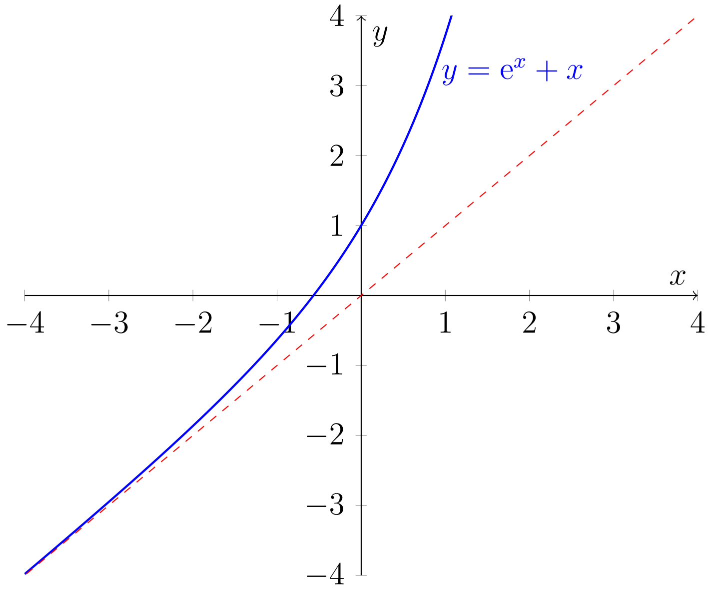
$$\lim_{x\to-\infty} [(\mathrm{e}^x + x) - x] = \lim_{x\to-\infty} \mathrm{e}^x = 0$$

Take the first and second derivatives.

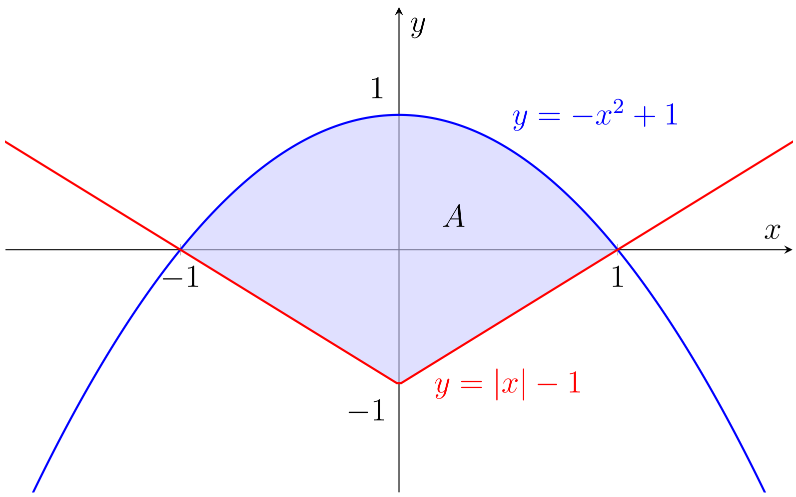
$$y' = \mathrm{e}^x + 1, \quad y'' = \mathrm{e}^x$$

We see that there are no critical or inflection points either. Now, set up a table and see what the graph looks like.

x	$(-\infty, \infty)$
y	$(-\infty, \infty)$
y' sign	$+$
y'' sign	$+$



6.



The area of the region is as follows.

$$A = \int_{-1}^{1} \left[(-x^2+1) - (|x|-1)\right]\, dx = \int_{-1}^{0} (-x^2+x+2)\, dx + \int_{0}^{1} (-x^2-x+2)\, dx$$

7. We'll use integration by parts.

$$\left.\begin{array}{l} \ln x = u \;\rightarrow\; \dfrac{1}{x}\, dx = du \\ x\, dx = dv \;\rightarrow\; \dfrac{x^2}{2} = v \end{array}\right\} \qquad \begin{aligned} \mathrm{I} &= \int x \ln x\, dx = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\, dx \\ &= \frac{x^2}{2} \cdot \ln x - \int \frac{x}{2}\, dx = \boxed{\frac{x^2}{2} \cdot \ln x - \frac{x^2}{4} + c,\; c \in \mathbb{R}} \end{aligned}$$

2021-2022 Fall Midterm (08/12/2021) Solutions
(Last update: 21/08/2025 16:06)

1.

(a) Determine one-sided limits.

$$\lim_{x\to-2^+}\frac{|x+2|}{|x|-2}=\lim_{x\to-2^+}\frac{x+2}{-x-2}=-1,\qquad \lim_{x\to-2^-}\frac{|x+2|}{|x|-2}=\lim_{x\to-2^-}\frac{-(x+2)}{-x-2}=1$$

$$\lim_{x\to-2^+}\frac{|x+2|}{|x|-2}\neq\lim_{x\to-2^-}\frac{|x+2|}{|x|-2}\implies \boxed{\text{The limit does not exist at } x=-2.}$$

(b) We can separate the limit into two limits because we are going to demonstrate that each limit exists individually.

$$\lim_{x\to0}\frac{x^4\sin(1/x)}{\sin(x^2)}=\lim_{x\to0}\frac{x^2}{\sin(x^2)}\cdot\lim_{x\to0}x^2\sin\left(\frac{1}{x}\right)\tag{16}$$

The left-hand limit in (1) is a standard form. Recall $\lim_{u\to0}\frac{\sin u}{u}=0$.

$$\lim_{x\to0}\frac{x^2}{\sin(x^2)}=\lim_{x\to0}\frac{1}{\frac{\sin(x^2)}{x^2}}=\frac{1}{\lim_{x\to0}\frac{\sin(x^2)}{x^2}}=\frac{1}{1}=1$$

The right-hand limit in (1) can be evaluated using the squeeze theorem. The trigonometric function $\sin\left(\frac{1}{x}\right)$ is continuous everywhere except at $x=0$.

$$-1\le\sin\left(\frac{1}{x}\right)\le1$$

$$-x^2\le x^2\sin\left(\frac{1}{x}\right)\le x^2$$

$$\lim_{x\to0}-x^2=\lim_{x\to0}x^2=0\implies\lim_{x\to0}x^2\sin\left(\frac{1}{x}\right)=0$$

Plug the values of the limits into (1) and find the result.

$$\lim_{x\to0}\frac{x^2}{\sin(x^2)}\cdot\lim_{x\to0}x^2\sin\left(\frac{1}{x}\right)=1\cdot0=\boxed{0}$$

(c) Notice that this is the definition of the derivative at some point. Let $f(x)=\arccos x$. Then $f'(x)=\dfrac{-1}{\sqrt{1-x^2}}$.

$$\lim_{x\to0}=\frac{\arccos x-\frac{\pi}{3}}{x-\frac{1}{2}}=\lim_{x\to0}\frac{f(x)-f\left(\frac{1}{2}\right)}{x-\frac{1}{2}}=f'\left(\frac{1}{2}\right)=\frac{-1}{\sqrt{1-\left(\frac{1}{2}\right)^2}}=\boxed{-\frac{2\sqrt{3}}{3}}$$

(d) Expand the expression by multiplying and dividing by its conjugate.

$$\lim_{x\to-\infty} x+\sqrt{x^2-x-4} = \lim_{x\to-\infty}\left[\left(x+\sqrt{x^2-x-4}\right)\cdot\frac{x-\sqrt{x^2-x-4}}{x-\sqrt{x^2-x-4}}\right]$$

$$= \lim_{x\to-\infty}\frac{x^2-(x^2-x-4)}{x-\sqrt{x^2-x-4}} = \lim_{x\to-\infty}\frac{x+4}{x-\sqrt{x^2-x-4}}$$

$$= \lim_{x\to-\infty}\frac{x\left(1+\frac{4}{x}\right)}{x-|x|\sqrt{1-\frac{1}{x}-\frac{4}{x^2}}} = \lim_{x\to-\infty}\frac{x\left(1+\frac{4}{x}\right)}{x\left(1+\sqrt{1-\frac{1}{x}-\frac{4}{x^2}}\right)}$$

$$= \lim_{x\to-\infty}\frac{1+\frac{4}{x}}{1+\sqrt{1-\frac{1}{x}-\frac{4}{x^2}}} = \frac{1+0}{1+\sqrt{1-0-0}} = \boxed{\frac{1}{2}}$$

2. Let r, h represent the radius and height of the cylinder, respectively, for both sections. The volume of the cylinder is given by

$$V = \pi r^2 h$$

The formula for V is continuous and bounded when written as a function of height. According to the extreme value theorem, extrema must exist on the boundary or at critical points.

(a) If this cylinder is vertically placed inside the hemisphere, using the Pythagorean theorem, we get the following relationship.

$$r^2+h^2=3^2 \implies r^2 = 9-h^2$$

Rewrite the volume formula by eliminating r.

$$V(h) = \pi\cdot h\cdot(9-h^2) = \pi\left(9h-h^3\right) \implies 0\le h\le 3$$

To maximize the volume, we need to determine the extrema of V by taking the first derivative.

$$V'(h) = \pi\left(9-3h^2\right) = 0 \implies 3h^2 = 9 \implies h=\sqrt{3}$$

Check the endpoints.

$$V(0)=0, \quad V(3)=0$$

Since $h=\sqrt{3}$, $r=\sqrt{9-\left(\sqrt{3}\right)^2}=\sqrt{6}$. The volume is then

$$V=\pi r^2 h = \pi\left(\sqrt{6}\right)^2\cdot\sqrt{3} = \boxed{6\pi\sqrt{3}}$$

(b) If the cylinder is placed horizontally, using the Pythagorean theorem, we get the following relationship.

$$\left(\frac{h}{2}\right)^2 + (2r)^2 = 3^2 \implies r^2 = \frac{1}{4}\left(9 - \frac{h^2}{4}\right)$$

Rewrite the volume formula.

$$V(h) = \pi \cdot \frac{1}{4}\left(9 - \frac{h^2}{4}\right) \cdot h \implies 0 \leq h \leq 6$$

Take the first derivative and find the extrema.

$$V'(h) = \frac{\pi}{4}\left(9 - \frac{3h^2}{4}\right) = 0 \implies h = 2\sqrt{3}$$

Check the endpoints.

$$V(0) = 0, \quad V(6) = 0$$

Since $h = 2\sqrt{3}$, $r = \sqrt{\frac{1}{4}\left(9 - \frac{(2\sqrt{3})^2}{4}\right)} = \frac{\sqrt{6}}{2}$. The volume is then

$$V = \pi r^2 h = \pi \cdot \left(\frac{\sqrt{6}}{2}\right)^2 \cdot 2\sqrt{3} = \boxed{3\pi\sqrt{3}}$$

3. Let $V(t)$, $h(t)$ represent the volume and height of the gas in the cylinder as a function of time, respectively. The volume of a right circular cylinder can be expressed as follows.

$$V(t) = \pi \cdot r^2 \cdot h(t)$$

Take the derivative of both sides with respect to t.

$$V'(t) = \pi r^2 \cdot h'(t)$$

It is given that $2r = 10$ cm, $h'(t) = -7$ cm/s. Therefore, $r = 5$ cm. The rate of change of volume at that moment is

$$V'(t) = 25\pi \cdot (-7) = \boxed{-175\pi \text{ cm}^2/\text{s}}$$

4.

(a) Let L be the value of the limit. Then we can take the logarithm of both sides. After that, we may take the logarithm of the expression inside the limit. The expression is continuous for $x \neq 0$.

$$L = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$$

$$\ln(L) = \ln\left[\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}\right] = \lim_{x\to 0}\ln\left[\left(\frac{\sin x}{x}\right)^{1/x^2}\right] = \lim_{x\to 0}\frac{\ln\left(\frac{\sin x}{x}\right)}{x^2}$$

Recall that $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$. Therefore, the limit is in the indeterminate form $0/0$. Apply L'Hôpital's rule where $0/0$ forms occur.

$$\ln(L) = \lim_{x\to 0}\frac{\ln\left(\frac{\sin x}{x}\right)}{x^2} \overset{\text{L'H.}}{=} \lim_{x\to 0}\frac{\frac{1}{\frac{\sin x}{x}}\cdot\frac{\cos x\cdot x - \sin x\cdot 1}{x^2}}{2x} = \lim_{x\to 0}\frac{x\cos x - \sin x}{2x^2\cdot\sin x} \qquad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 0}\frac{1\cdot\cos x + x(-\sin x) - \cos x}{4x\cdot\sin x + 2x^2\cdot\cos x} = \lim_{x\to 0}\frac{-\sin x}{4\sin x + 2x\cos x} \qquad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 0}\frac{-\cos x}{4\cos x + 2\cos x + 2x(-\sin x)} = \frac{-\cos 0}{4\cos 0 + 2\cos 0 - 2\cdot 0\cdot\sin 0} = -\frac{1}{6}$$

If $\ln(L) = -\dfrac{1}{6}$, then $\boxed{L = \mathrm{e}^{-\frac{1}{6}}}$.

(b)

$$\frac{x}{y} = \cos(\pi xy) \implies x = y\cos(\pi xy)$$

Differentiate each side.

$$\begin{gathered}
1 = y'\cdot\cos(\pi xy) + y\cdot(-\sin(\pi xy))\cdot\pi(1\cdot y + xy') \\
1 = y'\cos(\pi xy) - \pi y^2\sin(\pi xy) - \pi xyy'\sin(\pi xy) \\
1 + \pi y^2\sin(\pi xy) = y'\left[\cos(\pi xy) - \pi xy\sin(\pi xy)\right]
\end{gathered}$$

$$y' = \frac{1 + \pi y^2\sin(\pi xy)}{\cos(\pi xy) - \pi xy\sin(\pi xy)}$$

Calculate y' at the point $(-1, 1)$.

$$y' = \frac{1 + \pi\cdot\sin(-\pi)}{\cos(-\pi) + \pi\sin(-\pi)} = -1$$

Recall the tangent line formula: $y - y_0 = m(x - x_0)$, where m is basically the derivative at the point $(-1, 1)$. The tangent line is as follows.

$$y - 1 = -1(x + 1) \implies \boxed{y = -x}$$

5. The expression is defined $\forall x \in \mathbb{R} - \{-1\}$. Let us find the limit at infinity and the limit at negative infinity.

$$\lim_{x\to\infty} \frac{2x^2}{(x+1)^2} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{4x}{2(x+1)} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{4}{2} = 2 \qquad \lim_{x\to-\infty} \frac{2x^2}{(x+1)^2} = 2$$

The *only* horizontal asymptote is $y = 2$ and the *only* vertical asymptote is $x = -1$.

Take the first derivative.

$$y' = \frac{4x \cdot (x+1)^2 - 2x^2 \cdot 2(x+1)}{(x+1)^4} = \frac{4x}{(x+1)^3}$$

The *only* critical point occurs at $x = 0$.

Take the second derivative.

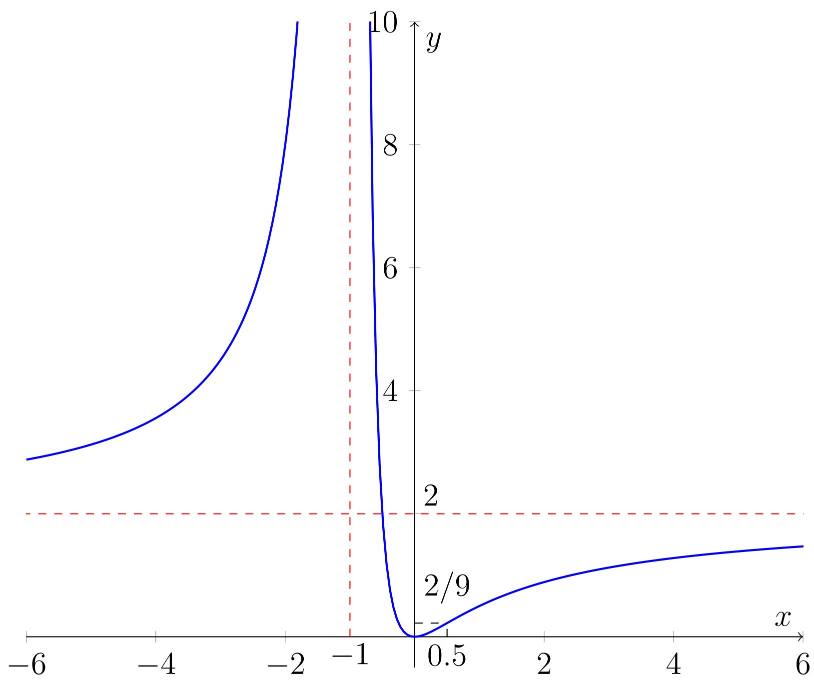
$$y'' = \frac{4 \cdot (x+1)^3 - 4x \cdot 3(x+1)^2}{(x+1)^6} = \frac{4-8x}{(x+1)^4}$$

The *only* inflection point occurs at $x = \dfrac{1}{2}$.

Eventually, consider some values of the function, such as the x- and y-intercepts, and set up a table.

$$f(0) = 0, \quad f\left(\frac{1}{2}\right) = \frac{2}{9}$$

x	$(-\infty,-1)$	$(-1,0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},\infty)$
y	$(2,\infty)$	$(0,\infty)$	$(0,\frac{2}{9})$	$(\frac{2}{9},2)$
y' sign	+	-	+	+
y'' sign	+	+	+	-



6. g is continuous and differentiable everywhere since it is a polynomial.

Since $g(-1) = -2 < 0$ and $g(0) = 1 > 0$, by IVT, there is at least one $c_1 \in (-1, 0)$ such that $g(c_1) = 0$.

Assume that g has another root c_2, i.e., $g(c_2) = 0$. Since g is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2), by MVT, there exists a $d \in (c_1, c_2)$ such that

$$g'(d) = \frac{g(c_2) - g(c_1)}{c_2 - c_1} = 0$$

But, we have

$$g'(x) = 7x^6 + 3x^2 + 1 \geq 0 + 0 + 1 = 1 > 0$$

So, $g'(x)$ cannot be 0. This yields a contradiction. Therefore, g has only one root.

2022-2023 Fall Midterm (23/11/2022) Solutions
(Last update: 29/08/2025 22:59)

1.

(a) Find the one-sided limits.

$$\lim_{x\to 0^+}\frac{|\sin x|}{x}=\lim_{x\to 0^+}\frac{\sin x}{x}=1,\qquad \lim_{x\to 0^-}\frac{|\sin x|}{x}=\lim_{x\to 0^-}\left(-\frac{\sin x}{x}\right)=-\lim_{x\to 0^-}\frac{\sin x}{x}=-1$$

The one-sided limits are not equal. Therefore, the limit does not exist.

(b) We have $-1\le\cos(1/x)\le 1$ for all $x\in\mathbb{R}-\{0\}$. So, $\mathrm{e}^{-1}\le \mathrm{e}^{\cos(1/x)}\le \mathrm{e}^{1}$.

$$x\mathrm{e}^{-1}\le x\mathrm{e}^{\cos(1/x)}\le x\mathrm{e}\implies \lim_{x\to 0^+}x\mathrm{e}^{-1}=\lim_{x\to 0^+}x\mathrm{e}=0\implies \lim_{x\to 0^+}x\mathrm{e}^{\cos(1/x)}=\boxed{0}$$

By the squeeze theorem, the limit is 0.

(c)

$$L=\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1-\sin x}}{x^3}$$

$$=\lim_{x\to 0}\left(\frac{\sqrt{1+\tan x}-\sqrt{1-\sin x}}{x^3}\cdot\frac{\sqrt{1+\tan x}+\sqrt{1-\sin x}}{\sqrt{1+\tan x}+\sqrt{1-\sin x}}\right)$$

$$=\lim_{x\to 0}\frac{(1+\tan x)-(1-\sin x)}{x^3\cdot\left(\sqrt{1+\tan x}+\sqrt{1-\sin x}\right)}=\lim_{x\to 0}\frac{\tan x+\sin x}{x^3\cdot\left(\sqrt{1+\tan x}+\sqrt{1-\sin x}\right)}$$

$$=\lim_{x\to 0}\frac{\sin x+\sin x\cdot\cos x}{\cos x\cdot x^3\cdot\left(\sqrt{1+\tan x}+\sqrt{1-\sin x}\right)}$$

$$=\lim_{x\to 0}\frac{1+\cos x}{\cos x\cdot x^2\cdot\left(\sqrt{1+\tan x}+\sqrt{1-\sin x}\right)}\cdot\lim_{x\to 0}\frac{\sin x}{x}\qquad\left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=\lim_{x\to 0}\left[\frac{1+\cos x}{\cos x\cdot x^2\cdot\left(\sqrt{1+\tan x}+\sqrt{1-\sin x}\right)}\cdot\frac{1-\cos x}{1-\cos x}\right]$$

$$=\lim_{x\to 0}\left[\frac{1-\cos^2 x}{\cos x\cdot x^2\cdot(1-\cos x)\cdot\left(\sqrt{1+\tan x}+\sqrt{1-\sin x}\right)}\right]\qquad\left[\sin^2 x+\cos^2 x=1\right]$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\lim_{x\to 0}\frac{1}{\cos x\cdot(1-\cos x)\cdot\left(\sqrt{1+\tan x}+\sqrt{1-\sin x}\right)}$$

$$=\lim_{x\to 0}\frac{1}{\cos x}\cdot\lim_{x\to 0}\frac{1}{1-\cos x}\cdot\lim_{x\to 0}\frac{1}{\sqrt{1+\tan x}+\sqrt{1-\sin x}}=1\cdot\lim_{x\to 0}\frac{1}{1-\cos x}\cdot\frac{1}{2}=\boxed{\infty}$$

2. Let $g(t)$, $f(t)$ represent the distance between point A and point B, and the distance between point A and the balloon, respectively. We may express the angle as follows.

$$\theta(t) = \arctan \frac{f(t)}{g(t)}$$

The first derivative of θ gives the rate of change of the angle. Apply the chain rule accordingly.

$$\theta'(t) = \frac{1}{1 + \dfrac{f^2(t)}{g^2(t)}} \cdot \frac{f'(t) \cdot g(t) - f(t) \cdot g'(t)}{g^2(t)} = \frac{f'(t) \cdot g(t) - f(t) \cdot g'(t)}{g^2(t) + f^2(t)}$$

At $t = t_0$, $f(t_0) = 200$, $g(t_0) = 100$, $f'(t_0) = 5$, $g'(t_0) = 0$. The reason why $g'(t_0) = 0$ is that the distance between the points does not change over time. Calculate $\theta'(t_0)$.

$$\theta'(t_0) = \frac{5 \cdot 100 - 200 \cdot 0}{100^2 + 200^2} = \frac{500}{50000} = \boxed{\frac{1}{100} \text{ rad/s}}$$

3.

(a) Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). The IVT states that since f is continuous on the interval, f takes any value on $[f(a), f(b)]$. The MVT states that since f is differentiable on the interval provided the continuity, there is at least one point such that the slope of the line that passes through the endpoints is equal to the slope of the line that is tangent to that point.

(b) Let $f(x) = x^{123} + 2x^{85} + 3x^{17} + 4x - 1$. Since this is a polynomial expression, f is continuous and differentiable everywhere. Arbitrarily choose $x = -1$ and $x = 1$ to make calculations easy. By IVT, f takes any value on $[f(-1), f(1)]$.

$$f(-1) = -11, \quad f(1) = 9$$

f must have at least one root x_1 on $[-1, 1]$ by IVT. Now, we need to prove that there is *only* one. We assume that there is another distinct root x_2. Since $f(x_1) = f(x_2) = 0$, at some point c, the first derivative of the function at this point is 0.

$$f'(c) = 123x^{122} + 170x^{84} + 51x^{16} + 4 \geq 0 + 0 + 0 + 4 = 4$$

$f'(c) > 0$. However, this is a contradiction. Therefore, there is *only* one root.

4. Differentiate both sides.

$$\frac{d}{dx}\left[\sin(y^2 e^{2x}) + \sqrt{\pi} y\right] = \frac{d}{dx}\left(x^2 + \pi\right)$$

$$\cos\left(y^2 e^{2x}\right) \cdot \left(2y\frac{dy}{dx}e^{2x} + y^2 e^{2x} \cdot 2\right) + \sqrt{\pi}\frac{dy}{dx} = 2x$$

$$2ye^{2x} \cos\left(y^2 e^{2x}\right) \frac{dy}{dx} + 2y^2 e^{2x} \cos\left(y^2 e^{2x}\right) + \sqrt{\pi}\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}\left[2y\mathrm{e}^{2x}\cos\left(y^2\mathrm{e}^{2x}\right)+\sqrt{\pi}\right]=2x-2y^2\mathrm{e}^{2x}\cos\left(y^2\mathrm{e}^{2x}\right)$$

$$\frac{dy}{dx}=\frac{2x-2y^2\mathrm{e}^{2x}\cos\left(y^2\mathrm{e}^{2x}\right)}{2y\mathrm{e}^{2x}\cos\left(y^2\mathrm{e}^{2x}\right)+\sqrt{\pi}} \tag{17}$$

Evaluate (1) at $(0,\sqrt{\pi})$.

$$\left.\frac{dy}{dx}\right|_{(0,\sqrt{\pi})}=\frac{2\cdot 0-2\left(\sqrt{\pi}\right)^2\mathrm{e}^0\cos\left(\left(\sqrt{\pi}\right)^2\mathrm{e}^0\right)}{2\sqrt{\pi}\mathrm{e}^0\cos\left(\left(\sqrt{\pi}\right)^2\mathrm{e}^0\right)+\sqrt{\pi}}=\frac{2\pi}{-\sqrt{\pi}}=-2\sqrt{\pi}$$

Use the straight line formula. $y-y_0=m(x-x_0)$, where $m=\left.\frac{dy}{dx}\right|_{(0,\sqrt{\pi})}$

$$\boxed{y=\sqrt{\pi}(1-2x)}$$

5. Let L be the value of the limit.

$$L=\lim_{x\to\infty}\left(\frac{\ln x}{x}\right)^{1/x}\qquad\left[\infty^0\right]$$

$$\ln(L)=\ln\left[\lim_{x\to\infty}\left(\frac{\ln x}{x}\right)^{1/x}\right]$$

Take the logarithm inside the limit because the expression is continuous for $x>0$.

$$\ln(L)=\lim_{x\to\infty}\ln\left[\left(\frac{\ln x}{x}\right)^{1/x}\right]=\lim_{x\to\infty}\left[\frac{\ln\left(\frac{\ln x}{x}\right)}{x}\right]\qquad\left[\frac{\infty}{\infty}\right]$$

$$\overset{\text{L'H.}}{=}\lim_{x\to\infty}\frac{\frac{1}{\frac{\ln x}{x}}\cdot\frac{\frac{1}{x}\cdot x-\ln x\cdot 1}{x^2}}{1}=\lim_{x\to\infty}\left(\frac{x}{\ln x}\cdot\frac{1-\ln x}{x^2}\right)=\lim_{x\to\infty}\frac{1-\ln x}{x\ln x}$$

$$=\lim_{x\to\infty}\frac{1}{x\ln x}-\lim_{x\to\infty}\frac{\ln x}{x\ln x}=0-\lim_{x\to\infty}\frac{1}{x}=0$$

If $\ln(L)=0$, then $\boxed{L=1}$.

6. First off, find the domain. The expression is undefined when the denominator is zero. Therefore, $(x-1)^2\neq 0\to x\neq 1$. The only vertical asymptote occurs at $x=1$.

$$\mathcal{D}=\mathbb{R}-\{1\}$$

Let us find the limit at infinity.

$$\lim_{x\to\infty}\frac{x^2-2}{(x-1)^2}\overset{\text{L'H.}}{=}\lim_{x\to\infty}\frac{2x}{2(x-1)}\overset{\text{L'H.}}{=}\lim_{x\to\infty}\frac{2}{2}=1$$

Similarly,

$$\lim_{x \to -\infty} \frac{x^2 - 2}{(x-1)^2} = 1$$

The horizontal asymptote occurs only at $y = 0$.

Take the first derivative by applying the quotient rule.

$$y' = \frac{(2x) \cdot (x-1)^2 - (x^2 - 2) \cdot 2(x-1)}{(x-1)^4} = \frac{4 - 2x}{(x-1)^3}$$

y' is undefined for $x = 1$, and $y' = 0$ for $x = 2$. Since 1 is not in the domain, the *only* critical point is $x = 2$.

Take the second derivative.

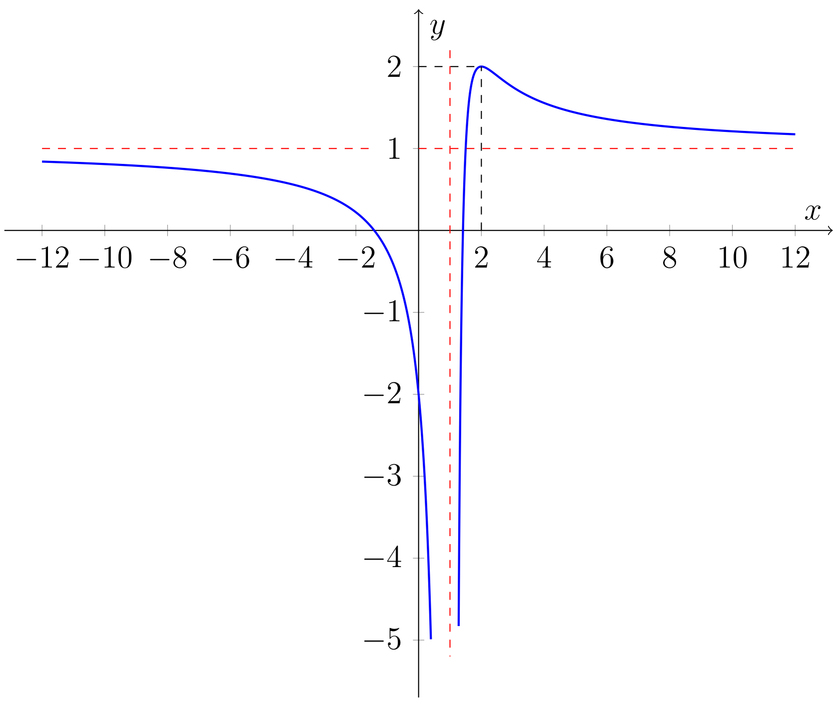
$$y'' = \frac{(-2) \cdot (x-1)^3 - (4 - 2x) \cdot 3(x-1)^2}{(x-1)^6} = \frac{4x - 10}{(x-1)^4}$$

The only inflection point occurs at $x = \dfrac{5}{2}$.

Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f\left(-\sqrt{2}\right) = f\left(\sqrt{2}\right) = 0,\ f(0) = -2,\ f(2) = 2,\ f(5/2) = 17/9$$

x	$(-\infty, -\sqrt{2})$	$(-\sqrt{2}, 0)$	$(0, 1)$	$(1, \sqrt{2})$	$(\sqrt{2}, 2)$	$(2, \frac{5}{2})$	$(\frac{5}{2}, \infty)$
y	$(0, 1)$	$(-2, 0)$	$(-\infty, -2)$	$(-\infty, 0)$	$(0, 2)$	$(\frac{17}{9}, 2)$	$(1, \frac{17}{9})$
y' sign	-	-	-	+	+	-	-
y'' sign	-	-	-	-	-	-	+



2023-2024 Fall Midterm (01/11/2023) Solutions
(Last update: 20/10/2025 17:04)

1.

(a)

$$\lim_{t\to 0}\frac{\tan^{-1}t}{\sin^{-1}t}=\lim_{t\to 0}\left(\frac{\tan^{-1}t}{\sin^{-1}t}\cdot\frac{t}{t}\right)=\lim_{t\to 0}\left(\frac{\tan^{-1}t}{t}\cdot\frac{1}{\frac{\sin^{-1}t}{t}}\right)$$

$$=\lim_{t\to 0}\frac{\tan^{-1}t}{t}\cdot\frac{1}{\lim\limits_{t\to 0}\dfrac{\sin^{-1}t}{t}}=1\cdot 1=\boxed{1}$$

The limits above are well-known limits. If we're supposed to write these limits in the form of $\sin x$ and x, follow these steps.

$$\lim_{t\to 0}\frac{\tan^{-1}t}{t}\overset{u=\tan^{-1}t}{=}\lim_{u\to 0}\frac{u}{\tan u}=\lim_{u\to 0}\frac{\cos u}{\frac{\sin u}{u}}=\frac{\lim\limits_{u\to 0}\cos u}{\lim\limits_{u\to 0}\dfrac{\sin u}{u}}=\frac{1}{1}=1$$

$$\lim_{t\to 0}\frac{\sin^{-1}t}{t}\overset{v=\sin^{-1}t}{=}\lim_{v\to 0}\frac{v}{\sin v}=\frac{1}{\lim\limits_{u\to 0}\dfrac{\sin v}{v}}=\frac{1}{1}=1$$

(b)

$$\begin{aligned}-1\le \sin(1/x)&\le 1\\ \mathrm{e}^{-1}\le \mathrm{e}^{\sin(1/x)}&\le \mathrm{e}^{1}\\ x\mathrm{e}^{-1}\le x\mathrm{e}^{\sin(1/x)}&\le x\mathrm{e}\\ \lim_{x\to 0^+}x\mathrm{e}^{-1}\le \lim_{x\to 0^+}x\mathrm{e}^{\sin(1/x)}&\le \lim_{x\to 0^+}x\mathrm{e}\\ 0\le \lim_{x\to 0^+}x\mathrm{e}^{\sin(1/x)}&\le 0\end{aligned}$$

By the squeeze theorem, the limit is $\boxed{0}$.

2. Let $V(t)$, $S(t)$, $r(t)$ represent the volume, surface area and radius, respectively. $r'(t)=2$ for all t. The rate of change of volume at $t=t_0$ is

$$V'(t_0)=4\pi r^2(t_0)r'(t_0)\qquad \left[V(t)=\frac{4}{3}\pi r^3(t)\right]$$

We also have $S(t_0)=4\pi$. Using the surface area formula $S(t)=4\pi r^2(t)$, we find that at $t=t_0$, the radius of the balloon is 1. Therefore, the rate of change of volume can now be evaluated.

$$V'(t_0)=4\pi\cdot 1^2\cdot 2=\boxed{8\pi\,\mathrm{cm}^3/\mathrm{s}}$$

3.

(a) The MVT states that if a function f is continuous on $[a, b]$ and differentiable on (a, b), there is at least one point P on the interval (a, b) such that the slope of the line that passes through the endpoints is equal to the slope of the line that is tangent to P.

(b) Let $f(x) = \mathrm{e}^x$. f is continuous on $[0, x]$ and differentiable on $(0, x)$. There exists at least one point c on $(0, x)$ such that

$$f'(c) = \mathrm{e}^c = \frac{\mathrm{e}^x - 1}{x} = \frac{f(x) - f(0)}{x - 0}$$

From the inequality $0 < c < x$,

$$\begin{gathered}
\mathrm{e}^0 < \mathrm{e}^c < \mathrm{e}^x \\
1 < \frac{\mathrm{e}^x - 1}{x} < \mathrm{e}^x \\
x < \mathrm{e}^x - 1 < x\mathrm{e}^x \\
1 + x < \mathrm{e}^x < 1 + x\mathrm{e}^x
\end{gathered}$$

4. Let us find the derivative of x^x with respect to x.

$$\begin{gathered}
y = x^x \\
\ln(y) = \ln(x^x) = x \ln x \\
\frac{1}{y} \cdot y' = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1 \\
y' = x^x(\ln x + 1)
\end{gathered}$$

We can now differentiate both sides of the equation given in the question.

$$\begin{gathered}
\frac{d}{dx}\left(y^2 x^x + xy\right) = \frac{d}{dx} 2 \\
2y \cdot y' \cdot x^x + y^2 \cdot x^x(\ln x + 1) + 1 \cdot y + x \cdot y' = 0 \\
y' \cdot (2y \cdot x^x + x) = -y - y^2 \cdot x^x(\ln x + 1)
\end{gathered}$$

$$y' = -\frac{y + y^2 \cdot x^x(\ln x + 1)}{2y \cdot x^x + x}$$

Evaluating y' at $(1, 1)$ gives $-\frac{2}{3}$. Using the straight line formula $y - y_0 = m(x - x_0)$, we get

$$\boxed{y - 1 = -\frac{2}{3}(x - 1)}$$

5. Take the logarithm of both sides of the equation and apply L'Hôpital's rule.

$$\ln(5) = \ln\left[\lim_{x\to\infty}\left(\frac{1+\dfrac{A}{x}}{1-\dfrac{2A}{x}}\right)^x\right] = \lim_{x\to\infty}\ln\left[\left(\frac{1+\dfrac{A}{x}}{1-\dfrac{2A}{x}}\right)^x\right] = \lim_{x\to\infty}\left[x\ln\left(\frac{1+\dfrac{A}{x}}{1-\dfrac{2A}{x}}\right)\right]$$

$$= \lim_{x\to\infty}\left\{x\left[\ln\left(1+\frac{A}{x}\right)-\ln\left(1-\frac{2A}{x}\right)\right]\right\} \quad [\infty\cdot 0]$$

$$= \lim_{x\to\infty}\frac{\ln\left(1+\dfrac{A}{x}\right)-\ln\left(1-\dfrac{2A}{x}\right)}{\dfrac{1}{x}} \quad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to\infty}\left[\frac{\left(\dfrac{1}{1+\frac{A}{x}}\right)\cdot\left(-\dfrac{A}{x^2}\right)-\left(\dfrac{1}{1-\frac{2A}{x}}\right)\cdot\left(\dfrac{2A}{x^2}\right)}{-\dfrac{1}{x^2}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\left(-\dfrac{A}{x^2}\right)\cdot\left(\dfrac{1}{1+\frac{A}{x}}+\dfrac{2}{1-\frac{2A}{x}}\right)}{-\dfrac{1}{x^2}}\right]$$

$$= A\lim_{x\to\infty}\frac{1}{\frac{A}{x}+1} + A\lim_{x\to\infty}\frac{2}{1-\frac{2A}{x}} = A\cdot 1 + A\cdot 2 = 3A$$

If $\ln(5) = 3A$, then $\boxed{A = \dfrac{\ln 5}{3}}$.

6. First off, find the domain. The expression is defined *only* for $x > 0$. The only vertical asymptote occurs at $x = 0$.

$$\mathcal{D} = \mathbb{R}^+$$

The limits as $x \to \infty$ and as $x \to 0^+$ are:

$$\lim_{x\to\infty}(\ln x)^2 = \infty, \quad \lim_{x\to 0^+}(\ln x)^2 = \infty$$

Take the first derivative to find the critical points.

$$y' = 2\ln x \cdot \frac{1}{x}$$

The *only* critical point is $x = 1$.

Take the second derivative by applying the quotient rule.

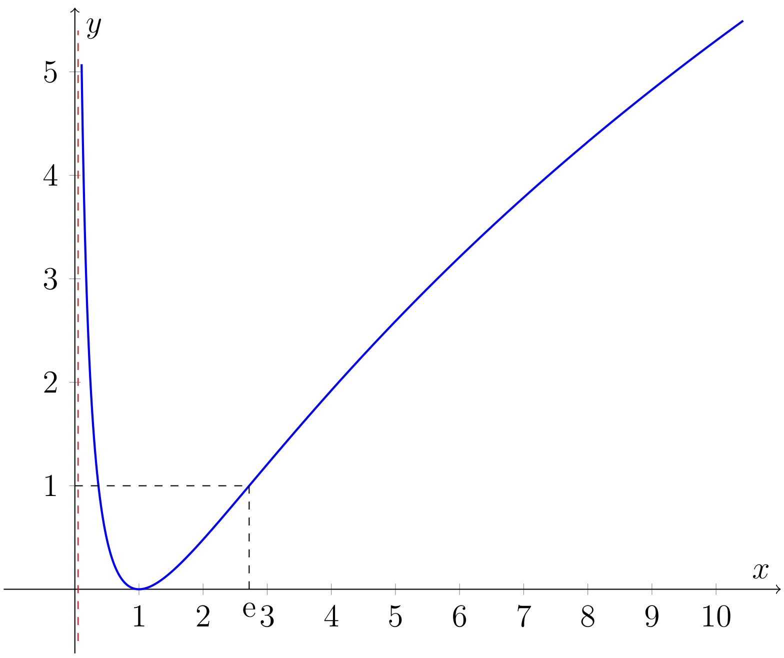
$$y'' = 2 \cdot \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \frac{2 - 2\ln x}{x^2}$$

The *only* inflection point is $x = \mathrm{e}$.

Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f(1) = 0,\ f(\mathrm{e}) = 1$$

x	$(0,1)$	$(1,\mathrm{e})$	(e,∞)
y	$(0,\infty)$	$(0,1)$	$(1,\infty)$
y' sign	-	+	+
y'' sign	+	+	-



2024-2025 Fall Midterm (02/12/2024) Solutions
(Last update: 06/11/2025 23:23)

1. To ensure continuity at $x = 0$, the one-sided limit values must be equal to the value of the function at that point.

$$\lim_{x\to 0^-} \frac{\tan ax}{\tan bx} = \lim_{x\to 0^+} (ax+b) = f(0) = 4$$

The easy part is that we can calculate the limit from the right.

$$\lim_{x\to 0^+} (ax+b) = 0 + b = b$$

Hence, $b = 4$. To calculate from the left, we need another technique.

$$\lim_{x\to 0^-} \frac{\tan ax}{\tan bx} = \lim_{x\to 0^-} \left(\frac{\sin ax}{\cos ax} \cdot \frac{\cos bx}{\sin bx} \cdot \frac{bx}{bx} \cdot \frac{ax}{ax} \right)$$

$$= \lim_{x\to 0^-} \left(\frac{\sin ax}{ax} \right) \cdot \lim_{x\to 0^-} \left(\frac{1}{\frac{\sin bx}{bx}} \right) \cdot \lim_{x\to 0^-} \left(\frac{\cos(bx)\cdot ax}{\cos(ax)\cdot bx} \right)$$

$$= 1 \cdot \frac{1}{\lim\limits_{x\to 0^-} \frac{\sin bx}{bx}} \cdot \lim_{x\to 0^-} \left(\frac{\cos(bx)\cdot a}{\cos(ax)\cdot b} \right) = 1 \cdot 1 \cdot \left(\frac{\cos(0)\cdot a}{\cos(0)\cdot b} \right)$$

$$= \frac{a}{b}$$

Now, set $\frac{a}{b} = b \implies a = 16$. $\boxed{a = 16,\ b = 4}$

2. Let $f(x) = x^{1/3}$ and $g(x) = x^{1/2}$. Using the differential approximation, we get

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x = x^{1/3} + \frac{1}{3}x^{-2/3}\Delta x$$
$$g(x+\Delta x) \approx g(x) + g'(x)\Delta x = x^{1/2} + \frac{1}{2}x^{-1/2}\Delta x$$

Set $x = 64$ and $\Delta x = 2$.

$$3\sqrt[3]{66} + 2\sqrt{66} \approx 3\left(64^{1/3} + \frac{1}{3}\cdot 64^{-2/3}\cdot 2\right) + 2\left(64^{1/2} + \frac{1}{2}\cdot 64^{-1/2}\cdot 2\right)$$
$$= 3\left(4 + \frac{1}{24}\right) + 2\left(8 + \frac{1}{8}\right) = \boxed{28.375}$$

3.

(a) Factor the numerator and use the conjugate of the expression $\cos x - 1$.

$$\lim_{x\to 0} \frac{5 - 6\cos x + \cos^2 x}{x \sin x} = \lim_{x\to 0} \frac{(\cos x - 1)(\cos x - 5)}{x \sin x}$$

$$= \lim_{x\to 0} \frac{(\cos x - 1)(\cos x - 5)(\cos x + 1)}{(x\sin x)(\cos x + 1)} = \lim_{x\to 0} \left(-\frac{\sin^2 x \cdot (\cos x - 5)}{x \sin x \cdot (\cos x + 1)} \right)$$

$$= \lim_{x\to 0} \left(-\frac{\sin x \cdot (\cos x - 5)}{x \cdot (\cos x + 1)} \right) = -\lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\cos x - 5}{\cos x + 1} = -1 \cdot \frac{\cos 0 - 5}{\cos 0 + 1} = \boxed{2}$$

(b) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x + 3| < \delta \implies |f(x) - 1| < \epsilon$.

$$|f(x) - 1| = \left|\sqrt{-x-2} - 1\right| = \left| \left(\sqrt{-x-2} - 1\right) \cdot \frac{\sqrt{-x-2} + 1}{\sqrt{-x-2} + 1} \right|$$
$$= \left| \frac{-x-3}{\sqrt{-x-2} + 1} \right| \le \frac{|-x-3|}{1} = |x+3| \qquad \left[\sqrt{-x-2} + 1 \ge 0 + 1 = 1\right]$$

We need to ensure that for all $\epsilon > 0$, there exists such a δ satisfying the inequality. To control the expression $\sqrt{-x-2} + 1$ in the denominator, we can assume that $|x - 3| < 1$. Then, the inequality $\sqrt{-x-2} + 1 \ge 1$ holds. We need to guarantee $\delta = 1$ when $\epsilon > 1$ because of the restriction $|x - 3| < 1$. Therefore, let $\delta = \min(1, \epsilon)$.

$$\left|\sqrt{-x-2} - 1\right| \le |x - 3| < \delta \le \epsilon$$

(c) Let L be the value of the limit. Then, take the logarithm of both sides. Since the expression is continuous for $x > 1$, we can take the logarithm function inside the limit.

$$L = \lim_{x\to 1^+} \left(\sqrt{x}\right)^{\ln(x-1)} \implies \ln(L) = \ln\left[\lim_{x\to 1^+} \left(\sqrt{x}\right)^{\ln(x-1)}\right]$$

$$\ln(L) = \lim_{x\to 1^+} \ln\left[(\sqrt{x})^{\ln(x-1)}\right] = \lim_{x\to 1^+} \left[\ln(x-1) \cdot \ln(\sqrt{x})\right] = \lim_{x\to 1^+} \frac{\ln(x-1)}{\frac{1}{\ln(\sqrt{x})}} \quad \left[\frac{\infty}{\infty}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 1^+} \frac{\frac{1}{x-1}}{\frac{1}{-\ln^2(\sqrt{x})} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}} = \lim_{x\to 1^+} \frac{\ln^2(\sqrt{x}) \cdot 2x}{1 - x} \quad \left[\frac{0}{0}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 1^+} \frac{2\ln\left(\sqrt{x}\right) \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \cdot 2x + \ln^2(\sqrt{x}) \cdot 2}{-1} = \lim_{x\to 1^+} \left[-2\ln\left(\sqrt{x}\right) - 2\ln^2\left(\sqrt{x}\right)\right]$$

$$= 2\ln\left(\sqrt{1}\right) + 2\ln^2\left(\sqrt{1}\right) = 0$$

$\ln(L) = 0$. Therefore, $\boxed{L = 1}$.

4. Let $f(x)$ represent the volume of coffee in the cone in cubic inches. The coffee in the cone will have a conical shape while draining. We may set up the equation below using the formula of the volume of a cone.

$$f(t) = \frac{1}{3} \cdot h(t) \cdot \pi r^2(t)$$

$h(t), r(t)$ represent the height and radius of the circular area that coffee forms, respectively, in inches. We can eliminate r to proceed with h. r and h are proportional.

$$\frac{r}{h} = \frac{2}{5} \implies r = \frac{2h}{5}$$

$$f(t) = \frac{4\pi h^3(t)}{75}$$

Take the derivative of both sides.

$$f'(t) = \frac{4\pi}{25} \cdot h^2(t) \cdot h'(t)$$

Given that at $t = t_0$, $f'(t_0) = -2.25$, $h(t_0) = 3$. We may now find $h'(t_0)$. Solve for $h'(t_0)$.

$$h'(t_0) = \frac{25 f'(t_0)}{4\pi h^2(t_0)} = \frac{25 \cdot (-2.25)}{4\pi \cdot (3)^2} = \boxed{-\frac{1.5625}{\pi} \text{ inches/minute}}$$

5. Let $f(x) = \ln(1+x) - x$. We have $f(0) = \ln(1+0) - 0 = 0$. The mean value theorem (MVT) states that if a function $g(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point c such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}$$

f is continuous on $[0, x]$ and differentiable on $(0, x)$. By MVT, $\dfrac{f(x) - f(0)}{x - 0} = f'(c)$ provided for some point c such that $0 < c < x$.

$$f'(c) = \frac{1}{c+1} - 1 = \frac{\ln(x+1) - x}{x} = \frac{f(x) - f(0)}{x - 0}$$

$$\frac{1}{c+1} = \frac{\ln(x+1)}{x} \implies c + 1 = \frac{x}{\ln(x+1)}$$

$$c = \frac{x - \ln(x+1)}{\ln(x+1)}$$

From the inequality $0 < c < x$,

$$0 < \frac{x - \ln(x+1)}{\ln(x+1)}$$

$$0 < x - \ln(x+1)$$

$$\ln(x+1) < x$$

6.

(a) First off, find the domain. The expression is undefined when the denominator is zero. Therefore, $(x-1)^2 \neq 0 \implies x \neq 1$. The only vertical asymptote occurs at $x = 1$.

$$\mathcal{D} = \mathbb{R} - \{1\}$$

Let us find the limit at infinity.

$$\lim_{x\to\infty} \frac{x^2-2}{(x-1)^2} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{2x}{2(x-1)} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{2}{2} = 1$$

Similarly,

$$\lim_{x\to-\infty} \frac{x^2-2}{(x-1)^2} = 1$$

The horizontal asymptote occurs only at $y = 0$.

Take the first derivative by applying the quotient rule.

$$y' = \frac{(2x)\cdot(x-1)^2 - (x^2-2)\cdot 2(x-1)}{(x-1)^4} = \frac{4-2x}{(x-1)^3}$$

y' is undefined for $x = 1$, and $y' = 0$ for $x = 2$. Since 1 is not in the domain, the *only* critical point is $x = 2$.

Take the second derivative.

$$y'' = \frac{(-2)\cdot(x-1)^3 - (4-2x)\cdot 3(x-1)^2}{(x-1)^6} = \frac{4x-10}{(x-1)^4}$$

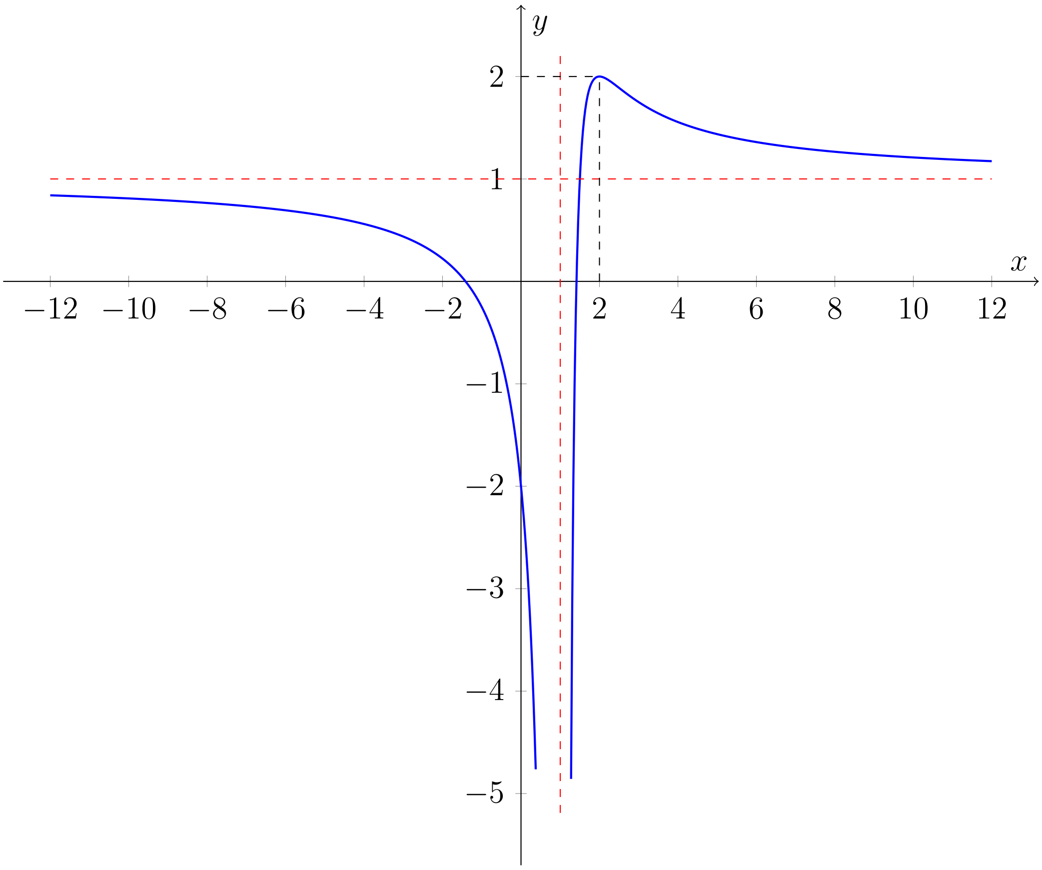
The only inflection point occurs at $x = \dfrac{5}{2}$.

(b) Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f\left(-\sqrt{2}\right) = f\left(\sqrt{2}\right) = 0,\ f(0) = -2,\ f(2) = 2,\ f(5/2) = 17/9$$

x	$\left(-\infty,-\sqrt{2}\right)$	$\left(-\sqrt{2},0\right)$	$(0,1)$	$\left(1,\sqrt{2}\right)$	$\left(\sqrt{2},2\right)$	$\left(2,\frac{5}{2}\right)$	$\left(\frac{5}{2},\infty\right)$
y	$(1,0)$	$(-2,0)$	$(-\infty,-2)$	$(-\infty,0)$	$(0,2)$	$\left(2,\frac{17}{9}\right)$	$\left(\frac{17}{9},1\right)$
y' sign	-	-	-	+	+	-	-
y'' sign	-	-	-	-	-	-	+

(c)



FINAL SOLUTIONS

2011-2012 Spring Final (29/05/2012) Solutions
(Last update: 30/08/2025 00:25)

1. y is implicitly defined as a function of x. Differentiate each side and solve for $\dfrac{dy}{dx}$.

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(4(x+1)^3\right) \implies 2y\frac{dy}{dx} = 12(x+1)^2 \implies \frac{dy}{dx} = \frac{6(x+1)^2}{y}$$

Since we're interested in the upper part of the curve (i.e., $y > 0$), $y = 2(x+1)^{3/2}$.

$$\frac{dy}{dx} = \frac{6(x+1)^2}{2(x+1)^{3/2}} = 3\sqrt{x+1}$$

The length of a curve defined by $y = f(x)$ whose derivative is continuous on the interval $a \le x \le b$ can be evaluated using the integral

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Set $a = 0,\ b = 1,\ \dfrac{dy}{dx} = 3\sqrt{x+1}$ and find the length.

$$S = \int_0^1 \sqrt{1 + \left(3\sqrt{x+1}\right)^2}\, dx = \int_0^1 \sqrt{9x+10}\, dx$$

Let $u = 9x + 10$, then $du = 9\, dx$.

$$S = \int_0^1 \sqrt{9x+10}\, dx = \int \frac{1}{9}\sqrt{u}\, du = \frac{1}{9}\cdot\frac{2}{3}u^{3/2} + c = \frac{2}{27}\left(9x+10\right)^{3/2}\Big|_0^1$$

$$= \boxed{\frac{2}{27}\left(19^{3/2} - 10^{3/2}\right)}$$

2. The derivative of f^{-1} at a point can be calculated using the rule

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Find the point where $f(x) = 4$. We could intuitively say $f(1) = 4$ because $f(1) = 1 + 2\cdot 1^3 + 1^3 = 4$. Therefore,

$$\left(f^{-1}\right)'(4) = \frac{1}{f'\left(f^{-1}(4)\right)} = \frac{1}{f'(1)}$$

Calculate the derivative of f at the point $x = 1$.

$$f'(x) = 1 + 4x + 3x^2 \implies f'(1) = 1 + 4\cdot 1 + 3\cdot 1^2 = 8$$

So,

$$\left(f^{-1}\right)'(4) = \boxed{\frac{1}{8}}$$

3.

(a)

$$\mathrm{I} = \int \cos^3 x \sin^2 x \, dx \qquad \left[\sin^2 + \cos^2 x = 1\right]$$

$$= \int \cos x \cdot \left(1 - \sin^2 x\right) \cdot \sin^2 x \, dx$$

Let $u = \sin x$, then $du = \cos x \, dx$.

$$\mathrm{I} = \int \cos x \cdot \left(1 - \sin^2 x\right) \cdot \sin^2 x \, dx = \int \left(1 - u^2\right) u^2 \, du = \int \left(u^2 - u^4\right) \, du = \frac{u^3}{3} - \frac{u^5}{5} + c$$

$$= \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c}$$

(b) Let $x = 4\sin u$, then $dx = 4\cos u \, du$ for $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

$$\mathrm{I} = \int \frac{x^2}{\sqrt{16 - x^2}} \, dx = \int \frac{16\sin^2 u}{\sqrt{16 - 16\sin^2 u}} \cdot 4\cos u \, du = \int \frac{16\sin^2 u \cos u}{|\cos u|} \, du \ [\cos u > 0]$$

$$= \int 16\sin^2 u \, du = 16 \int \left(1 - \cos^2 u\right) \, du = 16 \int \frac{1 - \cos 2u}{2} \, du = 8\left(u - \frac{\sin 2u}{2}\right) + c$$

$$= 8u - 8\sin u \cos u + c$$

Recall: $x = 4\sin u$. Then

$$x^2 = 16\sin^2 u \implies x^2 = 16 - 16\cos^2 u \implies \cos^2 u = \frac{16 - x^2}{16} \implies \cos u = \sqrt{1 - \frac{x^2}{16}}$$

$$\sin u = \frac{x}{4} \implies u = \arcsin \frac{x}{4}$$

Rewrite the integral.

$$\mathrm{I} = \boxed{8\arcsin \frac{x}{4} - 2x\sqrt{1 - \frac{x^2}{16}} + c, \quad c \in \mathbb{R}}$$

(c) Use the method of partial fraction decomposition.

$$\mathrm{I} = \int \frac{x^3-1}{x^3-x}\,dx = \int \frac{(x-1)\left(x^2+x+1\right)}{x(x-1)(x+1)}\,dx = \int \frac{x^2+x+1}{x^2+x}\,dx = \int \left(1+\frac{1}{x^2+x}\right)dx$$

$$= \int dx + \int \frac{1}{x(x+1)}\,dx = x + \int \left(\frac{A}{x}+\frac{B}{x+1}\right)dx$$

$$\begin{aligned} A(x+1)+B(x) &= 1 \\ x(A+B)+A &= 1 \\ \therefore A+B=0 \quad &[\text{eliminate}\,x] \;\to\; A=1 \implies B=-1 \end{aligned}$$

$$\mathrm{I} = x + \int \left(\frac{A}{x}+\frac{B}{x+1}\right)dx = x + \int \left(\frac{1}{x}-\frac{1}{x+1}\right)dx = \boxed{x+\ln|x|-\ln|x+1|+c,\ c\in\mathbb{R}}$$

(d) Use the method of integration by parts.

$$\left.\begin{aligned} u=\ln x &\implies du=\frac{1}{x}\,dx \\ dv=x^{123}\,dx &\implies v=\frac{x^{124}}{124} \end{aligned}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\mathrm{I} = \ln x \cdot \frac{x^{124}}{124} - \int \frac{x^{124}}{124}\cdot\frac{1}{x}\,dx = \boxed{\frac{\ln x \cdot x^{124}}{124} + \frac{x^{124}}{124^2} + c, \quad c\in\mathbb{R}}$$

4. Let L be the value of the limit.

$$L = \lim_{x\to\infty} (\ln x)^{\frac{1}{x}}$$

Take the logarithm of both sides. We can take the logarithm inside the limit because the expression is continuous for $x > 0$. After that, apply L'Hôpital's rule where $0/0$ or ∞/∞ forms occur.

$$\ln(L) = \ln\left(\lim_{x\to\infty}(\ln x)^{\frac{1}{x}}\right) = \lim_{x\to\infty} \ln\left[\ln(x)^{\frac{1}{x}}\right] = \lim_{x\to\infty}\frac{\ln(\ln x)}{x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$\overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x\ln x} = 0$$

Since $\ln(L) = 0$, $\boxed{L=1}$.

5.

(a) Since the numerator is constant, we can take it out of the summation.

$$\sum_{n=1}^{\infty} \frac{5}{3^n} = 5 \cdot \sum_{n=1}^{\infty} \frac{1}{3^n} = 5 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

This is a geometric series with $r = \dfrac{1}{3} < 1$. Therefore, the series converges.

(b) Take the limit of the sequence at infinity. We can take the limit inside the trigonometric function because it is continuous everywhere.

$$\lim_{n\to\infty} \cos\left(\frac{1}{5^n}\right) = \cos\left(\lim_{n\to\infty} \frac{1}{5^n}\right) = \cos(0) = 1 \neq 0$$

By the nth Term Test for divergence, the series diverges.

(c) Let $a_n = \dfrac{n!}{\mathrm{e}^{2\mathrm{n}}}$. Then,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{\mathrm{e}^{2(n+1)}} \cdot \frac{\mathrm{e}^{2n}}{n!}\right| = \lim_{n\to\infty} \left|\frac{n+1}{\mathrm{e}^2}\right| = \infty > 1$$

By the Ratio Test, the series diverges.

(d) Let $a_n = f(n)$, where $n \in \mathbb{N}$. The function $f(x) = \dfrac{x}{\mathrm{e}^{x^2}}$ is positive, continuous and decreasing for $x > 1$.

$$\left.\begin{array}{r} x > 0 \\ \mathrm{e}^{x^2} > 0 \end{array}\right\} \text{ for } x > 1 \implies \frac{x}{\mathrm{e}^{x^2}} > 0$$

e^{x^2} grows at a higher rate than x. Therefore, f is decreasing. The expressions are continuous for $x > 1$. We may now apply the Integral Test. Handle the improper integral with the limit.

$$\int_1^{\infty} \frac{x}{\mathrm{e}^{x^2}}\,dx = \lim_{R\to\infty} -\frac{1}{2}\mathrm{e}^{-x^2}\Big|_1^R = \lim_{R\to\infty} -\frac{1}{2}\left(\mathrm{e}^{-R^2} - \mathrm{e}^{-1}\right) = \frac{1}{2\mathrm{e}} \quad \text{(converges)}$$

By the Integral Test, the series converges.

(Bonus)

(a) Let $y = f(x)$ be a continuous function on a bounded interval, and let f be differentiable on the same interval except possibly at the endpoints. Then $f'(x)$ gives the first derivative. $f'(x)$ gives the instantaneous rate of change of the function at a certain point and it gives the slope of the line that is tangent to the graph of the function at that point.

(b) Given $f(t)$ describes the displacement, $f'(t)$ corresponds to the instantaneous velocity of the object.

2012-2013 Fall Final (23/01/2013) Solutions
(Last update: 30/08/2025 00:29)

1.

(a) Evaluate the limit of the non-alternating part at infinity.

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) \overset{n=\frac{1}{u}}{=} \lim_{u\to 0}\frac{\sin u}{u} = 1$$

However, for odd values of n, the limit at infinity becomes negative. On the other hand, for even values of n, the limit at infinity becomes positive. Therefore, the limit at infinity does not exist. So, the sequence diverges.

(b) The exponential function e^x and the trigonometric function $\cos x$ are continuous everywhere.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \mathrm{e}^{\cos\left(\frac{1}{n}\right)} = \mathrm{e}^{\cos\left(\lim\limits_{n\to\infty}\frac{1}{n}\right)} = \mathrm{e}^{\cos 0} = \mathrm{e}^1 = \mathrm{e}$$

The sequence converges to $\boxed{\mathrm{e}}$.

2.

(a) Let $a_n = f(n)$. Define $f(x) = \dfrac{x}{(3+x^2)^{3/4}}$. The function is continuous for $x > 1$ because the numerator and the denominator are continuous for $x > 1$ and $(3+x^2)^{3/4} \neq 0,\ \forall x \in \mathbb{R}$.

$$\left.\begin{array}{c} x > 0 \\ (3+x^2)^{3/4} > 0 \end{array}\right\} \text{ for } x > 1 \implies \frac{x}{(3+x^2)^{3/4}} > 0$$

For $x > 1$, $x < x^{3/2} = (x^2)^{3/4} < (3+x^2)^{3/4}$. The denominator grows faster than the numerator. Therefore, the function is decreasing.

We may now apply the Integral Test. Handle the improper integral by taking the limit.

$$\int_1^\infty \frac{x}{(3+x^2)^{3/4}}\,dx = \lim_{R\to\infty} 2\left(3+x^2\right)^{1/4}\Big|_1^R = 2\lim_{R\to\infty}\left[(3+R^2)^{1/4} - \left(3+1^2\right)^{1/4}\right] = \infty$$

Since the integral diverges, the series also diverges.

(b) $\ln(1+x) < x$ for $x > -1$. Therefore,

$$\frac{1}{\sqrt{n}}\ln\left(\frac{n+1}{n-1}\right) = \frac{1}{\sqrt{n}}\ln\left(1+\frac{2}{n-1}\right) < \frac{1}{\sqrt{n}}\cdot\frac{2}{n-1}$$

Since $n \geq 2$, we have the inequality $2n-2 \geq n \implies \dfrac{2}{n} \geq \dfrac{1}{n-1}$.

$$\frac{1}{\sqrt{n}} \cdot \frac{2}{n-1} \leq \frac{1}{\sqrt{n}} \cdot \frac{4}{n} = \frac{4}{n^{3/2}}$$

Now, let $a_n = \frac{1}{\sqrt{n}} \ln\left(\frac{n+1}{n-1}\right)$ and $b_n = \frac{4}{n^{3/2}}$. Apply the Direct Comparison Test.

$$0 < a_n < b_n \implies 0 < \frac{1}{\sqrt{n}} \ln\left(\frac{n+1}{n-1}\right) < \frac{4}{n^{3/2}}$$

b_n converges by the p-series test because $3/2 > 1$. Since b_n converges, by the Direct Comparison Test, a_n also converges.

(c) Apply the Ratio Test. Let $a_n = \frac{(2n)!}{5^n (n!)^2}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2(n+1))!}{5^{n+1}\left((n+1)!\right)^2} \cdot \frac{5^n (n!)^2}{(2n)!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(2n+2)\cdot(2n+1)\cdot((2n)!)\cdot(n!)^2\cdot 5^n}{5^n \cdot 5 \cdot (n+1)^2 \cdot (n!)^2 \cdot ((2n)!)}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(2n+2)(2n+1)}{5(n+1)^2}\right| = \lim_{n\to\infty} \left|\frac{4n^2+6n+2}{5n^2+10n+5}\right|$$

Now, take the corresponding function and evaluate the limit using L'Hôpital's rule.

$$\lim_{x\to\infty} \left|\frac{4x^2+6x+2}{5x^2+10x+5}\right| \overset{\text{L'H.}}{=} \lim_{x\to\infty} \left|\frac{8x+6}{10x+10}\right| \overset{\text{L'H.}}{=} \lim_{x\to\infty} \left|\frac{8}{10}\right| = \frac{4}{5} < 1$$

By the Ratio Test, the series converges absolutely. Since the series converges absolutely, the series converges.

(d) Let $a_n = \frac{1}{n^2} \mathrm{e}^{1/n}$. Define $f(x) = \frac{1}{x^2} \mathrm{e}^{1/x}$. The function is continuous for $x > 1$ because the expressions $\frac{1}{x^2}$ and $\mathrm{e}^{1/x}$ are continuous for $x > 1$ and $x^2 \neq 0$, $\forall x > 1$.

$$\left.\begin{array}{r} x^2 > 0 \\ \mathrm{e}^{1/x} > 0 \end{array}\right\} \text{ for } x > 1 \implies \frac{\mathrm{e}^{1/x}}{x^2} > 0$$

For $x > 1$, $\mathrm{e}^{1/x}$ tends to 1 and x^2 is increasing. Therefore, the function is decreasing for $x > 1$.

We may now apply the Integral Test. Handle the improper integral by taking the limit.

$$\int_1^\infty \frac{\mathrm{e}^{1/x}}{x^2}\, dx- = \lim_{R\to\infty} -\mathrm{e}^{1/x}\Big|_1^\infty = \lim_{R\to\infty} \left(-\mathrm{e}^{1/R} + \mathrm{e}^1\right) = \mathrm{e} - 1 \quad \text{(convergent)}$$

Since the integral converges, by the Integral Test, the series also converges.

3.

(a) Add and subtract e^x in the numerator.

$$\int \frac{dx}{\mathrm{e}^x+1} = \int \frac{1+\mathrm{e}^x-\mathrm{e}^x}{\mathrm{e}^x+1}\,dx = \int \frac{\mathrm{e}^x+1}{\mathrm{e}^x+1}\,dx - \int \frac{\mathrm{e}^x}{\mathrm{e}^x+1}\,dx$$

The result of the integral on the left is x. To calculate the integral on the right, use the u-substitution. Let $u=\mathrm{e}^x+1$, then $du=\mathrm{e}^x\,dx$.

$$x-\int \frac{du}{u} = x-\ln|u|+c = \boxed{x-\ln|\mathrm{e}^x+1|+c = x-\ln(\mathrm{e}^x+1)+c,\ c\in\mathbb{R}}\ [\mathrm{e}^x+1>0]$$

(b) Solve the integral using the method of integration by parts.

$$\left.\begin{aligned} u=\arcsin x &\implies du=\frac{1}{\sqrt{1-x^2}}\,dx \\ dv=x\,dx &\implies v=\frac{x^2}{2} \end{aligned}\right\} \to \int u\,dv = uv-\int v\,du$$

$$\int x\arcsin x\,dx = \frac{x^2}{2}\arcsin x - \int \frac{x^2}{2\sqrt{1-x^2}}\,dx \tag{18}$$

Now, we need to find the result of the integral on the right. Let $x=\sin u$ for $\frac{\pi}{2}<u<\frac{\pi}{2}$. Then $dx=\cos u\,du$.

$$\int \frac{x^2}{2\sqrt{1-x^2}}\,dx = \int \frac{\sin^2 u}{2\sqrt{1-\sin^2 u}}\cdot\cos u\,du = \int \frac{\sin^2 u\cos u}{2\sqrt{\cos^2 u}}\,du \quad [\sin^2 u+\cos^2 u=1]$$

$$= \int \frac{\sin^2 u\cos u}{2|\cos u|}\,du \quad [\cos u>0]$$

$$= \frac{1}{2}\int \sin^2 u\,du = \frac{1}{2}\int \left(1-\cos^2 u\right)\,du = \frac{1}{2}\int\left(\frac{1-\cos 2u}{2}\right)\,du$$

$$= \frac{u}{4}-\frac{\sin 2u}{8}+c = \frac{u}{4}-\frac{\sin u\cos u}{4}+c, \quad c\in\mathbb{R}$$

Recall the equation $x=\sin u$.

$$x=\sin u \implies \arcsin x = u$$
$$x=\sin u \implies x^2=\sin^2 u = 1-\cos^2 u \implies \cos u=\sqrt{1-x^2}$$

Rewrite the result of the last integral.

$$\int \frac{x^2}{2\sqrt{1-x^2}}\,dx = \frac{1}{4}\left(\arcsin x - x\sqrt{1-x^2}\right)$$

Rewrite (1).

$$\int x \arcsin x \, dx = \boxed{\frac{x^2}{2} \arcsin x - \frac{1}{4} \arcsin x + \frac{1}{4} x\sqrt{1-x^2} + c, \quad c \in \mathbb{R}}$$

4. The length of a curve defined by $y = f(x)$ whose derivative is continuous on the interval $a \le x \le b$ can be evaluated using the integral.

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$$

Find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx} \int_0^x \sqrt{\cos(4t)} \, dt$$

By the Fundamental Theorem of Calculus, $\dfrac{dy}{dx}$ can be rewritten as

$$\frac{dy}{dx} = \sqrt{\cos(4x)}$$

Set $a = 0$, $b = \dfrac{\pi}{8}$, $\dfrac{dy}{dx} = \sqrt{\cos(4x)}$ and then find the length.

$$S = \int_0^{\pi/8} \sqrt{1 + \left(\sqrt{\cos(4x)}\right)^2} \, dx = \int_0^{\pi/8} \sqrt{1 + \cos(4x)} \, dx \quad \left[\cos(4x) = 2\cos^2(2x) - 1\right]$$

$$= \int_0^{\pi/8} \sqrt{2\cos^2(2x)} \, dx = \sqrt{2} \int_0^{\pi/8} |\cos(2x)| \, dx \quad [\cos(2x) > 0]$$

$$= \sqrt{2} \int_0^{\pi/8} \cos(2x) \, dx = \frac{\sqrt{2}}{2} \sin(2x) \Big|_0^{\pi/8} = \frac{\sqrt{2}}{2} \left(\sin \frac{\pi}{4} - \sin 0\right) = \boxed{\frac{1}{2}}$$

5.

(a) Find the horizontal asymptotes.

$$\lim_{x \to \pm\infty} \frac{1}{x^2 - 4} = 0$$

Find the vertical asymptotes. The expression is undefined for $x = \pm 2$.

$$\lim_{x \to 2^+} \frac{1}{x^2 - 4} = \lim_{x \to -2^+} \frac{1}{x^2 - 4} = \infty$$

$$\lim_{x \to 2^-} \frac{1}{x^2 - 4} = \lim_{x \to -2^-} \frac{1}{x^2 - 4} = -\infty$$

$$\boxed{\text{The horizontal asymptote is } y = 0\text{. The vertical asymptotes are } x = \pm 2.}$$

(b) Compute the first derivative and set it to 0 to find the critical points. Apply the product rule appropriately.

$$f'(x) = \frac{1 \cdot (x^2 - 4) - x \cdot (2x)}{(x^2 - 4)^2} = -\frac{x^2 + 4}{(x^2 - 4)^2}$$

f is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$. Therefore,

$$\boxed{\text{f is decreasing everywhere except at the undefined points.}}$$

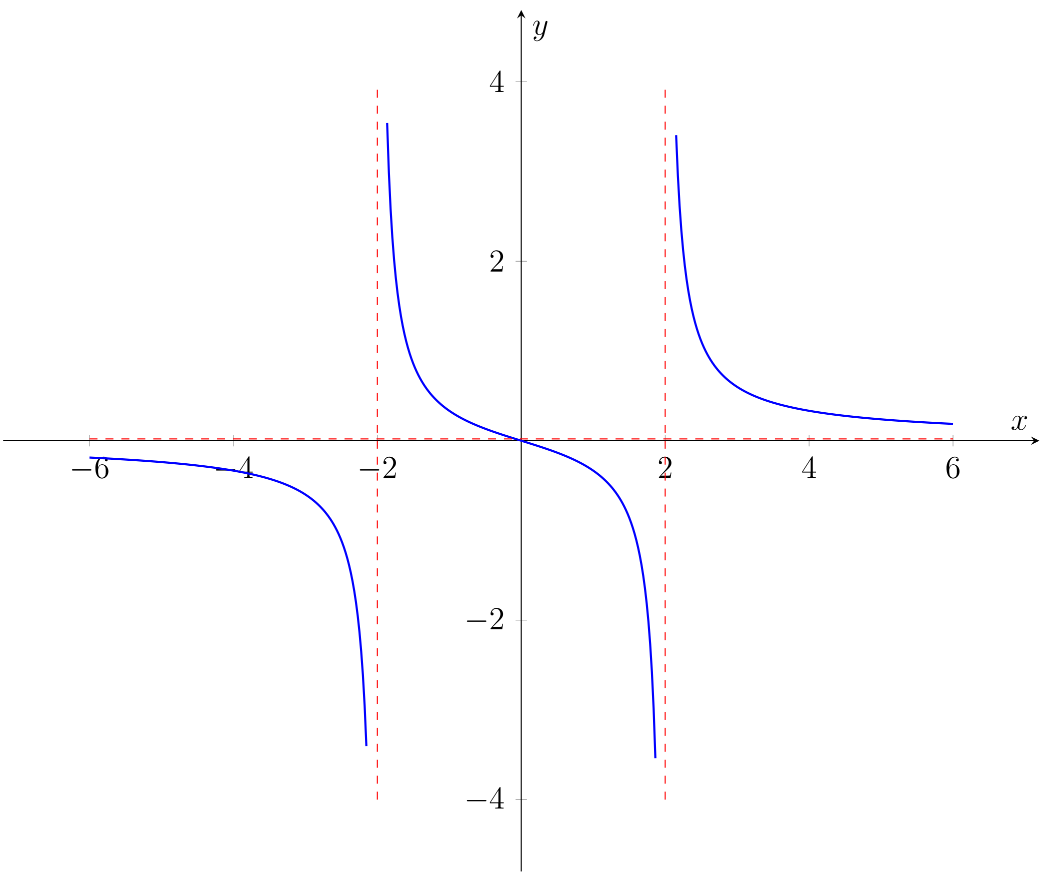
(c) $\boxed{\text{No local maximum or minimum values exist.}}$

(d) Compute the second derivative.

$$f''(x) = -\frac{2x \cdot (x^2 - 4)^2 - (x^2 + 4) \cdot 2 \cdot (x^2 - 4) \cdot (2x)}{(x^2 - 4)^4} = \frac{2x^3 + 24x}{(x^2 - 4)^3}$$

$$\boxed{\begin{array}{c}\text{An inflection point occurs at } x = 0. \\ \text{f is concave up for } -2 < x < 0 \cup x > 2. \text{ f is concave down for } x < -2 \cup 0 < x < 2.\end{array}}$$

(e)



2014-2015 Makeup (28/01/2015) Solutions
(Last update: 30/08/2025 00:40)

1. $\ln x$ is defined for $x > 0$. Therefore, $\ln(4 - x^2)$ is defined on $(-2, 2)$.

Let us find the limit as $x \to \pm 2$.

$$\lim_{x\to 2} \ln\left(4 - x^2\right) = \lim_{x\to -2} \ln\left(4 - x^2\right) = -\infty$$

The vertical asymptotes occur at $x = \pm 2$.

Take the first derivative and find the critical points. Apply the chain rule.

$$y' = \frac{d}{dx} \ln\left(4 - x^2\right) = \frac{1}{4 - x^2} \cdot (-2x) = -\frac{2x}{4 - x^2}$$

A critical point occurs at $x = 0$. At this point, the first derivative is 0.

Take the second derivative. Apply the quotient rule.

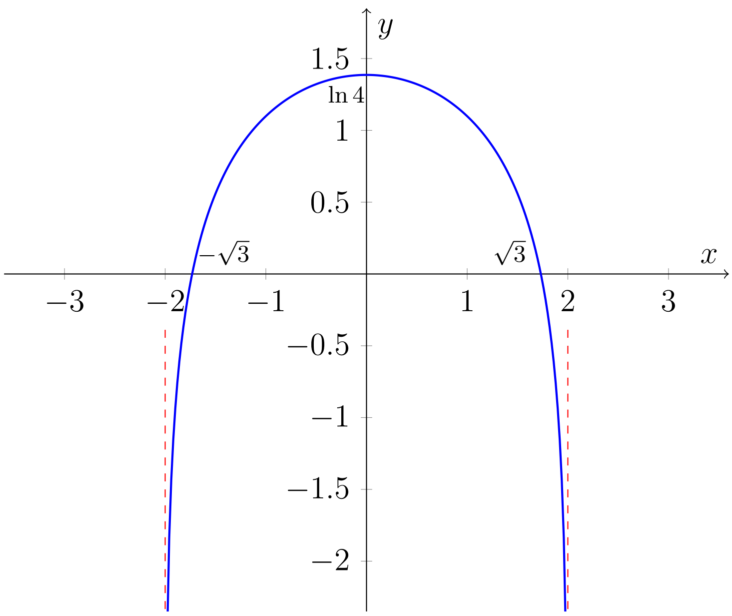
$$y'' = \frac{d}{dx}\left(\frac{-2x}{4 - x^2}\right) = \frac{-2 \cdot (4 - x^2) - (-2x) \cdot (-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

No inflection points occur.

Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f(0) = \ln 4$$

x	$(-2, 0)$	$(0, 2)$
y	$(-\infty, \ln 4)$	$(-\infty, \ln 4)$
y' sign	$+$	-
y'' sign	-	-



2. As $x \to \infty$, the expression is in the form $\infty \cdot 0$, which is an indeterminate form. Rationalize x.

$$\lim_{x\to\infty} x \arctan \frac{1}{x} = \lim_{x\to\infty} \frac{\arctan \frac{1}{x}}{\frac{1}{x}}$$

The limit is in the form $\frac{0}{0}$. L'Hôpital's rule suggests that we take the derivative of each side of the fraction if we encounter $\frac{0}{0}$ or $\frac{\infty}{\infty}$ forms.

$$\lim_{x\to\infty} \frac{\arctan \frac{1}{x}}{\frac{1}{x}} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1} = \boxed{1}$$

3. The manufacturer will minimize the use of material if the total surface is minimal. Let V be the volume of the can. The volume of the can may then be expressed as follows.

$$V = \pi r^2 h,$$

where r is the radius of the bottom and h is the height of the can. The total surface area of the can is

$$S = 2\pi h r + 2\pi r^2$$

Using the equalities $1\,\text{L} = 1\,\text{dm}^3$ and $V = 1500\,\text{L}$, rewrite h in terms of V.

$$V = 1500\,\text{L} = 1500\,\text{dm}^3 = \pi r^2 h \implies h = \frac{1500}{\pi r^2}$$

$$S = \frac{3000}{r} + 2\pi r^2$$

To minimize S, take the first derivative and set to 0 to find the critical points. Apply the quotient rule.

$$S' = -\frac{3000}{r^2} + 4\pi r = 0 \implies r^3 = \frac{3000}{4\pi} \implies r = \sqrt[3]{\frac{750}{\pi}}$$

Using the formula $h = \frac{V}{\pi r^2}$, find the height.

$$h = \frac{1500}{\pi \left(\frac{750}{\pi}\right)^{2/3}}$$

The dimensions for this can, in terms of the radius and height, are

$$\boxed{r = 5\sqrt[3]{\frac{6}{\pi}}\ \text{dm}, \quad h = 10\sqrt[3]{\frac{6}{\pi}}\ \text{dm}}$$

4. The length of a curve defined by $y = f(x)$ whose derivative is continuous on the interval $a \le x \le b$ can be evaluated using the integral

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$$

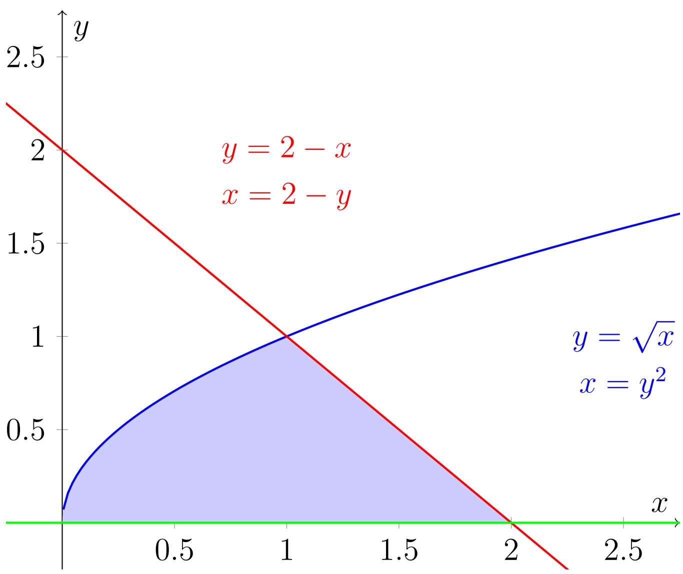
Set $a = 0,\ b = \pi/4$ and find the length.

$$S = \int_0^{\pi/4} \sqrt{1 + \left(\sqrt{-\tan x}\right)^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/4} |\sec x|\ dx = \int_0^{\pi/4} \sec x\, dx \qquad [\sec x > 0 \quad \text{for} \quad 0 < x \le \pi/4]$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/4} = \ln\left|\sqrt{2} + 1\right| - \ln 1 = \boxed{\ln\left(\sqrt{2} + 1\right)}$$

5.



(a)

$$V_{\text{Shell}} = \int_0^1 2\pi y \left[(2 - y) - (y^2)\right]\, dy$$

$$V_{\text{Washer}} = \int_0^1 \pi \left[\left(\sqrt{x}\right)^2 - (0)^2\right]\, dx + \int_1^2 \pi \left[(2 - x)^2 - (0)^2\right]\, dx$$

(b)

$$V_{\text{Shell}} = \int_0^1 2\pi x \left[\left(\sqrt{x}\right) - (0)\right] \, dx + \int_1^2 2\pi x \left[(2-x) - (0)\right] \, dx$$

$$V_{\text{Washer}} = \int_0^1 \pi \left[(2-y)^2 - \left(y^2\right)^2\right] \, dy$$

6. If the function $x = f(y) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the graph of $x = f(y)$ about the y-axis is

$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

We're interested in the right side of the parabola. So, let $f(y) = \sqrt{y}$ and set $a = 1$, $b = 4$. Find $\dfrac{dx}{dy}$ and then calculate the area.

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$S = \int_1^4 2\pi x \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2} \, dy = \int_1^4 2\pi \sqrt{y} \cdot \sqrt{1 + \frac{1}{4y}} \, dy = 2\pi \int_1^4 \sqrt{y + \frac{1}{4}} \, dy$$

$$= 2\pi \left[\frac{2}{3}\left(y + \frac{1}{4}\right)^{3/2}\right]_1^4 = \frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{5}{4}\right)^{3/2}\right] = \boxed{\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)}$$

7. Let the corresponding function be $f(x) = \dfrac{1}{x \ln x^2}$. f is decreasing because $\ln x^2$ and x are increasing for $x \geq 2$. These expressions are also continuous for $x \geq 2$. f is positive for $x \geq 2$. $x = 2 \implies \ln 2^2 = \ln 4 > \ln \mathrm{e} = 1 > 0$, $x = 2 > 0$. Since the conditions are satisfied, we may apply the Integral Test.

Let $u = \ln x^2$, then $du = \dfrac{1}{x^2} \cdot 2x \, dx \implies du = \dfrac{2}{x} \, dx$

$$x = 2 \implies u = \ln 2^2 = \ln 4, \qquad x \to \infty \implies u \to \infty$$

$$\int_2^\infty \frac{1}{x \ln x^2} \, dx = \int_{\ln 4}^\infty \frac{1}{2} \cdot \frac{1}{u} \, du = \lim_{R \to \infty} \int_{\ln 4}^R \frac{du}{2u} = \lim_{R \to \infty} \frac{1}{2} \ln u \bigg|_{\ln 4}^R$$

$$= \frac{1}{2} \lim_{R \to \infty} \left(\ln R - \ln(\ln 4)\right) = \boxed{\infty}$$

Since the integral diverges, the series also diverges.

8. Apply the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot(x-2)^{n+1}}{(n+1)\cdot(10)^{n+1}}\cdot\frac{n\cdot(10)^n}{(-1)^n\cdot(x-2)^n}\right| = \lim_{n\to\infty}\left|-\frac{n(x-2)}{10(n+1)}\right|$$

$$= \frac{|x-2|}{10}\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \frac{|x-2|}{10}$$

The series converges absolutely for $\dfrac{|x-2|}{10} < 1$.

$$\frac{|x-2|}{10} < 1 \implies |x-2| < 10 \implies -10 < x-2 < 10 \implies -8 < x < 12$$

Investigate the convergence at the endpoints (i.e., $x = -8$ and $x = 12$). Try $x = -8$.

$$x = -8 \to \sum_{n=1}^{\infty}(-1)^n\frac{1}{n\,10^n}(-10)^n = \sum_{n=1}^{\infty}(-1)^n\cdot(-1)^n\cdot\frac{1}{n} = \sum_{n=1}^{\infty}(-1)^{2n}\cdot\frac{1}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

For $x = -8$, we get a p-series where $p = 1$. By the p-series Test, it diverges. Try $x = 12$.

$$x = 12 \to \sum_{n=1}^{\infty}(-1)^n\frac{1}{n\,10^n}(10)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

This is an alternating series. The non-alternating part, which is $\frac{1}{n}$, is nonincreasing for $n \geq 2$ and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges.

The convergence set for the power series is

$$\boxed{(-8, 12]}$$

2014-2015 Summer Final (23/08/2015) Solutions
(Last update: 20/10/2025 23:04)

1. $\ln x$ is defined for $x > 0$. Therefore, $\ln(x^2+1)$ is defined on $\mathbb{R}$ because $x^2+1 \geq 1 > 0$.

Let us find the limit at infinity and negative infinity.

$$\lim_{x\to\infty} \ln\left(x^2+1\right) = \lim_{x\to-\infty} \ln\left(x^2+1\right) = \infty$$

No asymptotes occur.

Take the first derivative and find the critical points. Apply the chain rule.

$$y' = \frac{d}{dx}\ln\left(x^2+1\right) = \frac{1}{x^2+1}\cdot 2x = \frac{2x}{x^2+1}$$

A critical point occurs at $x = 0$. At this point, the first derivative is 0.

Take the second derivative. Apply the quotient rule.

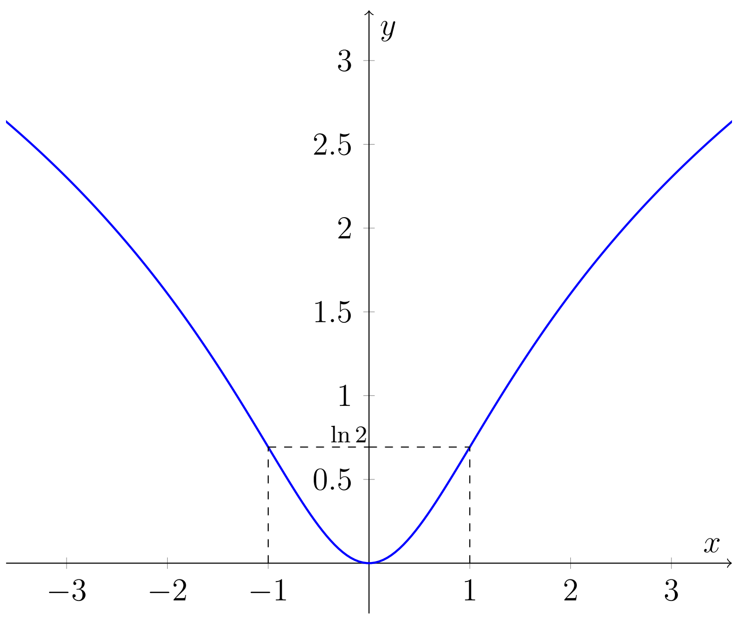
$$y'' = \frac{d}{dx}\left(\frac{2x}{x^2+1}\right) = \frac{2\cdot(x^2+1) - 2x\cdot(2x)}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$$

The inflection points occur at $x = \pm 1$. At these points, the direction of the curvature changes.

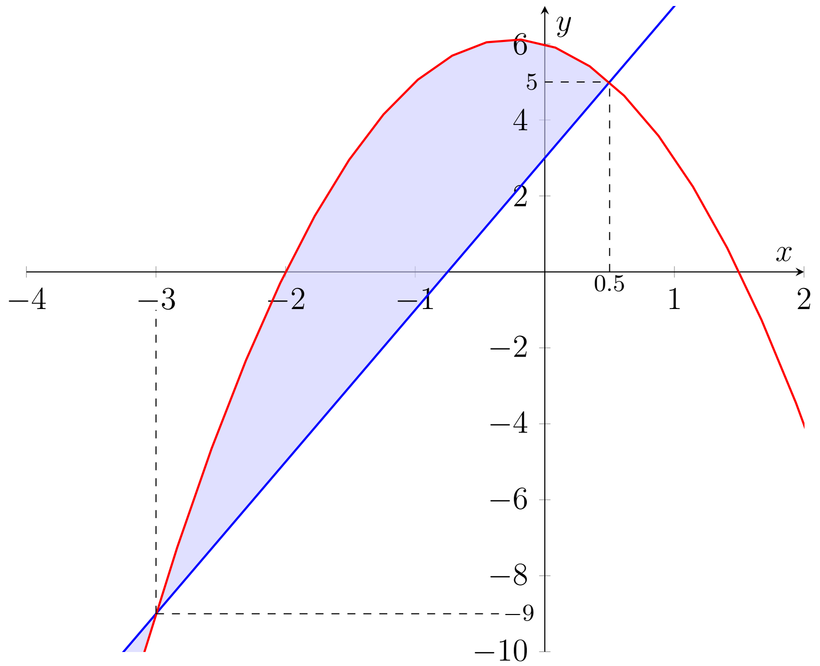
Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f(-1) = f(1) = \ln 2, \quad f(0) = 0$$

x	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
y	$(\ln 2,\infty)$	$(0,\ln 2)$	$(0,\ln 2)$	$(\ln 2,\infty)$
y' sign	-	-	+	+
y'' sign	-	+	+	-



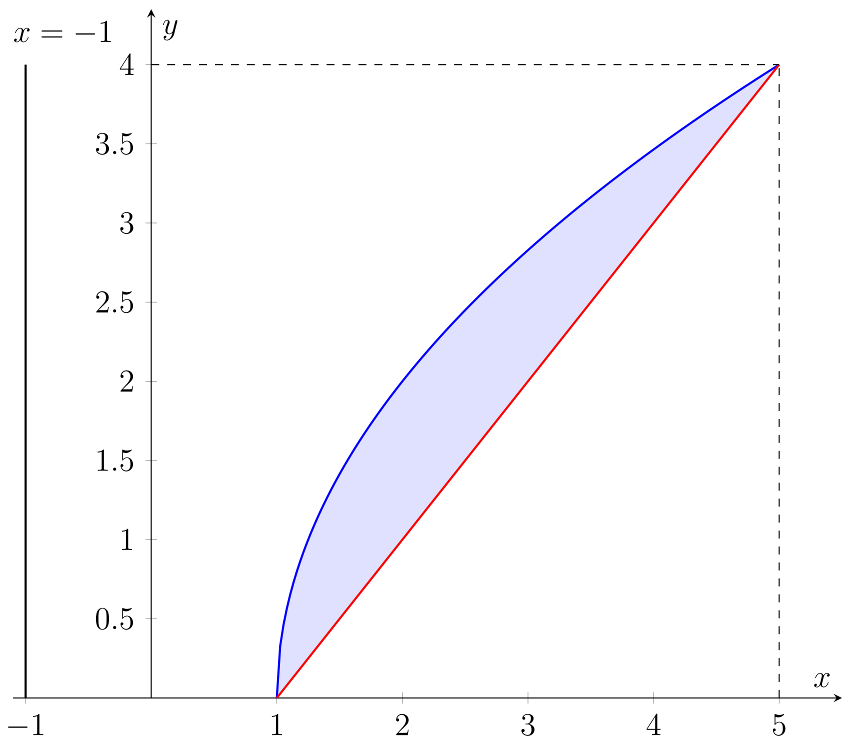
2.



$$\text{Area} = \int_{-3}^{1/2} \left[6 - x - 2x^2 - (4x+3)\right]\, dx = \int_{-3}^{1/2} \left(3 - 5x - 2x^2\right)\, dx$$

$$= 3x - \frac{5x^2}{2} - \frac{2x^3}{3}\Bigg|_{-3}^{1/2} = \left(\frac{3}{2} - \frac{5}{8} - \frac{1}{12}\right) - \left(-9 - \frac{45}{2} + 18\right) = \boxed{\frac{343}{24}}$$

3.



Rewrite the equations of the curves and solve for x. Let V be the volume of the solid.

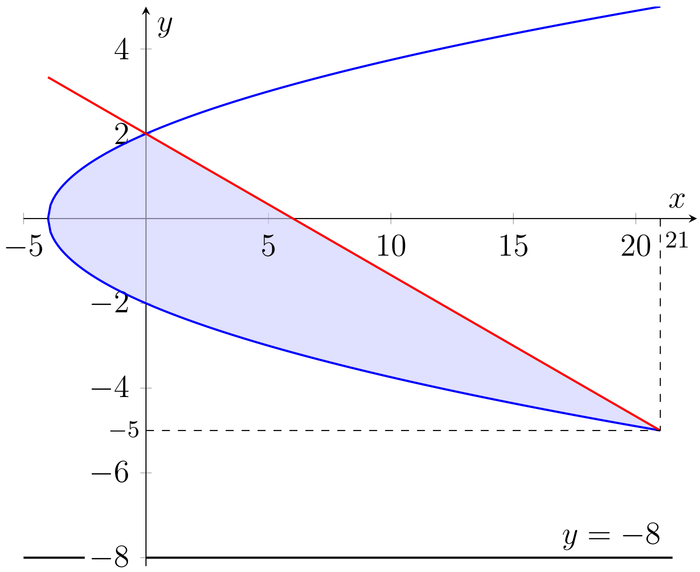
$$y = 2\sqrt{x-1} \implies y^2 = 4x - 4 \implies x = \frac{y^2+4}{4}$$
$$y = x - 1 \implies x = y + 1$$

$$V = \int_D \pi \left[R_2^2(y) - R_1^2(y)\right]\, dy = \int_0^4 \pi \left[((y+1)+1)^2 - \left(\left(\frac{y^2+4}{4}\right)+1\right)^2\right] dy$$

$$= \pi \int_0^4 \left[(y+2)^2 - \left(\frac{y^2}{4}+2\right)^2\right] dy = \pi \int_0^4 \left(y^2 + 4y + 4 - \frac{y^4}{16} - y^2 - 4\right) dy$$

$$= \pi \int_0^4 \left(4y - \frac{y^4}{16}\right) dy = \pi \left[2y^2 - \frac{y^5}{80}\right]_0^4 = \pi\left[\left(32 - \frac{64}{5}\right) - (0)\right] = \boxed{\frac{96\pi}{5}}$$

4.



Let V be the volume of the solid.

$$V = \int_D 2\pi \cdot h(y) \cdot r(y)\, dy = 2\pi \int_{-5}^{2} (y+8) \cdot \left[(6-3y) - (y^2-4)\right] dy$$

$$= 2\pi \int_{-5}^{2} (y+8)(-y^2 - 3y + 10)\, dy = 2\pi \int_{-5}^{2} \left(-y^3 - 3y^2 + 10y - 8y^2 - 24y + 80\right) dy$$

$$= 2\pi \int_{-5}^{2} \left(-y^3 - 11y^2 - 14y + 80\right) dy = 2\pi \left[-\frac{y^4}{4} - \frac{11y^3}{3} - 7y^2 + 80y\right]_{-5}^{2}$$

$$= 2\pi\left[\left(-4 - \frac{88}{3} - 28 + 160\right) - \left(\frac{625}{4} + \frac{1375}{3} - 175 - 400\right)\right] = \boxed{\frac{4459\pi}{6}}$$

5.

(a) Use the u-substitution method. Let $u = \mathrm{e}^{-x} + \ln x$. Then $du = \left(-\mathrm{e}^{-x} + \dfrac{1}{x}\right) dx$.

$$\int 4\left(\frac{1}{x} - \mathrm{e}^{-x}\right) \cos\left(\mathrm{e}^{-x} + \ln x\right)\, dx = \int 4\cos u\, du = 4\sin u + c$$

$$= \boxed{4\sin\left(\mathrm{e}^{-x} + \ln x\right) + c,\ c \in \mathbb{R}}$$

(b) Decompose the fraction into multiple partial fractions. Let $A,\ B,\ C,\ D,\ E,\ F \in \mathbb{R}$.

$$\mathrm{I} = \int \frac{x^3 + 10x^2 + 3x + 36}{(x-1)^2\,\left(x^2+4\right)^2}\, dx = \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2}\right) dx$$

Let $N = x^3 + 10x^2 + 3x + 36$.

$$N = A\left(x^2+4\right)^2(x-1) + B\left(x^2+4\right)^2 + (Cx+D)\left(x^2+4\right)(x-1)^2 + (Ex+F)(x-1)^2$$

$$= \left(x^2+4\right)^2 [A(x-1)+B] + (x-1)^2\left[(Cx+D)\left(x^2+4\right) + Ex + F\right]$$

$$\begin{aligned} &= \left(x^4 + 8x^2 + 16\right)(Ax - A + B) + \left(x^2 - 2x + 1\right) \\ &\quad \cdot \left(Cx^3 + 4Cx + Dx^2 + 4D + Ex + F\right) \end{aligned}$$

$$\begin{aligned} &= Ax^5 - Ax^4 + Bx^4 + 8Ax^3 - 8Ax^2 + 8Bx^2 + 16Ax - 16A + 16B \\ &\quad + Cx^5 + 4Cx^3 + Dx^4 + 4Dx^2 + Ex^3 + Fx^2 - 2Cx^4 - 8Cx^2 - 2Dx^3 - 8Dx \\ &\quad - 2Ex^2 - 2Fx + Cx^3 + 4Cx + Dx^2 + 4D + Ex + F \end{aligned}$$

$$\begin{aligned} &= x^5(A+C) + x^4(-A+B+D-2C) + x^3(8A+4C+E-2D+C) \\ &\quad + x^2(-8A+8B+4D+F-8C-2E+D) + x(16A-8D-2F+4C+E) \\ &\quad - 16A + 16B + 4D + F \end{aligned}$$

Equate the coefficients of like terms.

$$\begin{aligned} A + C &= 0 \qquad (19) \\ -A + B + D - 2C &= 0 \\ 8A + 5C + E - 2D &= 1 \\ -8A + 8B + 5D - 8C + F - 2E &= 10 \\ 16A - 8D - 2F + 4C + E &= 3 \\ -16A + 16B + 4D + F &= 36 \end{aligned}$$

From (1), $A = -C$. Rewrite C in terms of A and rearrange the equations.

$$
\begin{aligned}
A+B+D&=0 \\
3A+E-2D&=1 \\
8B+5D+F-2E&=10 \\
12A-8D-2F+E&=3 \\
-16A+16B+4D+F&=36
\end{aligned} \tag{20}
$$

From (2), $A+B=-D$. Rewrite D in terms of A and B and rearrange the equations.

$$5A+2B+E=1 \tag{21}$$
$$-5A+3B+F-2E=10 \tag{22}$$
$$20A+8B-2F+E=3 \tag{23}$$
$$-20A+12B+F=36 \tag{24}$$

By using the couples (3) & (4) and (4) & (5), eliminate E.

$$\left.\begin{aligned} 5A+7B+F &= 12 \\ 35A+19B-3F &= 16 \\ -20A+12B+F &= 36 \end{aligned}\right\} \implies \left.\begin{aligned} 50A+40B &= 52 \\ -25A+55B &= 124 \end{aligned}\right\} \implies A=-\frac{14}{25},\ B=2$$

Therefore, $C=\frac{14}{25}$, $D=-\frac{36}{25}$. From (3), $E=-\frac{1}{5}$, and from (6), $F=\frac{4}{5}$.

Substitute the values into A, B, C, D, E, F.

$$\mathrm{I}=\int\left(-\frac{14}{25}\cdot\frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{\frac{14}{25}x-\frac{36}{25}}{x^2+4}+\frac{-\frac{1}{5}x+\frac{4}{5}}{\left(x^2+4\right)^2}\right)dx \tag{25}$$

From now on, integrate term by term. Integrate the first term in (7).

$$\int -\frac{14}{25}\cdot\frac{1}{x-1}\,dx=-\frac{14}{25}\int\frac{1}{x-1}\,dx=-\frac{14}{25}\ln|x-1|+c \tag{26}$$

Integrate the second term in (7).

$$\int\frac{2}{(x-1)^2}\,dx=-\frac{2}{x-1}+c \tag{27}$$

Integrate the third term in (7).

$$\int\frac{\frac{14}{25}x-\frac{36}{25}}{x^2+4}\,dx=\frac{1}{25}\int\frac{14x-36}{x^2+4}\,dx=\frac{7}{25}\int\frac{2x}{x^2+4}\,dx-\frac{36}{25}\int\frac{1}{x^2+4}\,dx$$

$$=\frac{7}{25}\ln\left|x^2+4\right|-\frac{36}{100}\int\frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$$

$$=\frac{7}{25}\ln\left(x^2+4\right)-\frac{36}{50}\arctan\left(\frac{x}{2}\right)+c \tag{28}$$

Integrate the last term in (7).

$$\int \frac{-\frac{1}{5}x+\frac{4}{5}}{\left(x^2+4\right)^2}\,dx = \frac{1}{5}\int \frac{4-x}{\left(x^2+4\right)^2}\,dx = \frac{4}{5}\int \frac{1}{\left(x^2+4\right)^2}\,dx - \frac{1}{5}\int \frac{x}{\left(x^2+4\right)^2}\,dx \tag{29}$$

First, solve the integral on the left in (11). Let $x = 2\tan u$, then $dx = 2\sec^2 u\,du$.

$$\int \frac{1}{\left(x^2+4\right)^2}\,dx = \int \frac{2\sec^2 u}{\left(4\tan^2 u+4\right)^2}du = \int \frac{2\sec^2 u}{16\sec^4 u}\,du = \frac{1}{8}\int \frac{1}{\sec^2 u}\,du$$

$$= \frac{1}{8}\int \cos^2 u\,du = \frac{1}{8}\int \left(\frac{1-\cos 2u}{2}\right)\,du = \frac{1}{8}\left(\frac{u}{2} - \frac{\sin 2u}{4}\right) + c$$

$$= \frac{u}{16} - \frac{\sin u\cos u}{16} + c$$

Since $x = 2\tan u$, $\tan u = \dfrac{x}{2}$

$$u = \arctan\frac{x}{2}, \quad \sin u = \frac{x}{\sqrt{x^2+4}}, \quad \cos u = \frac{2}{\sqrt{x^2+4}}$$

Rewrite the integral.

$$\int \frac{1}{\left(x^2+4\right)^2}\,dx = \frac{1}{16}\left(\arctan\frac{x}{2} - \frac{2x}{x^2+4}\right) + c \tag{30}$$

Now, solve the integral on the right in (11). Let $u = x^2+4$, then $du = 2x\,dx$.

$$\int \frac{x}{\left(x^2+4\right)^2}\,dx = \int \frac{du}{2u^2} = -\frac{1}{2u} + c = -\frac{1}{2\left(x^2+4\right)} + c \tag{31}$$

Rewrite the integral in (11) using (12) and (13).

$$\frac{4}{5}\int \frac{1}{\left(x^2+4\right)^2}\,dx - \frac{1}{5}\int \frac{x}{\left(x^2+4\right)^2}\,dx = \frac{1}{20}\left(\arctan\frac{x}{2} + \frac{2x+2}{x^2+4}\right) + c \tag{32}$$

Eventually, using (8), (9), (10) and (14), rewrite (7).

$$\mathrm{I} = \boxed{-\frac{14}{25}\ln|x-1| - \frac{2}{x-1} + \frac{7}{25}\ln\left(x^2+4\right) - \frac{67}{100}\arctan\frac{x}{2} + \frac{x+1}{10\left(x^2+4\right)} + c,\ c\in\mathbb{R}}$$

(c) Let $x = \dfrac{2}{5}\sec u$ for $0 \le u < \dfrac{\pi}{2}$, then $dx = \dfrac{2}{5}\sec u\tan u\,du$.

$$\mathrm{I} = \int \frac{\sqrt{25x^2-4}}{x}\,dx = \int \frac{\sqrt{4\sec^2 u - 4}}{\frac{2}{5}\sec u}\cdot\frac{2}{5}\sec u\tan u\,du \qquad \left[\tan^2 u + 1 = \sec^2 u\right]$$

$$\mathrm{I} = 2\int |\tan u| \tan u \, du \qquad [\tan u > 0]$$

$$= 2\int \tan^2 u \, du = 2\int \sec^2 u \, du - 2\int du = 2\tan u - 2u + c$$

Recall that $x = \dfrac{2}{5}\sec u$.

$$\sec u = \frac{5x}{2} \implies \sec^2 u = \frac{25x^2}{4} \implies \tan u = \frac{\sqrt{25x^2-4}}{2} \implies u = \arctan\left(\frac{\sqrt{25x^2-4}}{2}\right)$$

$$\int \frac{\sqrt{25x^2-4}}{x} \, dx = \boxed{\sqrt{25x^2-4} - 2\arctan\left(\frac{\sqrt{25x^2-4}}{2}\right) + c, \quad c \in \mathbb{R}}$$

(d) Use the method of integration by parts.

$$\left.\begin{array}{r} u = x^2 \implies du = 2x\,dx \\ dv = \cos(4x)\,dx \implies v = \dfrac{1}{4}\sin(4x) \end{array}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\mathrm{I} = x^2 \cdot \frac{1}{4}\sin(4x) - \int \frac{1}{4}\sin(4x) \cdot 2x\,dx = \frac{x^2}{4}\sin(4x) - \frac{1}{2}\int x\sin(4x)\,dx$$

Apply the same procedure.

$$\left.\begin{array}{r} w = x \implies dw = dx \\ dz = \sin(4x)\,dx \implies z = -\dfrac{1}{4}\cos(4x) \end{array}\right\} \to \int w\,dz = wz - \int z\,dw$$

$$\mathrm{I} = \frac{x^2}{4}\sin(4x) - \frac{1}{2}\left[\frac{-x}{4}\cos(4x) - \int -\frac{1}{4}\cos(4x)\,dx\right]$$

$$= \boxed{\frac{x^2}{4}\sin(4x) + \frac{x}{8}\cos(4x) - \frac{1}{32}\sin(4x) + c, \quad c \in \mathbb{R}}$$

(e)

$$\int \frac{dx}{2x^2-3x+2} = \int \frac{dx}{2x^2-3x+\frac{9}{8}+\frac{7}{8}} = \int \frac{dx}{\left(\sqrt{2}x - \frac{3\sqrt{2}}{4}\right)^2 + \frac{7}{8}}$$

$$= \frac{8}{7}\int \frac{dx}{\dfrac{8\left(\sqrt{2}x - \frac{3\sqrt{2}}{4}\right)^2}{7} + 1}$$

Let $u = \dfrac{2\sqrt{2}}{\sqrt{7}}\left(\sqrt{2}x - \dfrac{3\sqrt{2}}{4}\right)$, then $du = \dfrac{4}{\sqrt{7}}\,dx$.

$$\frac{8}{7}\int \frac{dx}{\dfrac{8\left(\sqrt{2}x - \frac{3\sqrt{2}}{4}\right)^2}{7} + 1^2} = \frac{8}{7}\int \frac{1}{u^2+1}\cdot\frac{\sqrt{7}}{4}\,du = \frac{2}{\sqrt{7}}\int \frac{1}{u^2+1}\,du = \frac{2}{\sqrt{7}}\arctan u + c$$

$$= \frac{2}{\sqrt{7}}\arctan\left[\frac{2\sqrt{2}}{\sqrt{7}}\left(\sqrt{2}x - \frac{3\sqrt{2}}{4}\right)\right] + c, \quad c \in \mathbb{R}$$

$$= \boxed{\frac{2}{\sqrt{7}}\arctan\left(\frac{4x-3}{\sqrt{7}}\right) + c, \quad c \in \mathbb{R}}$$

6. Use the method of integration by parts.

$$\left.\begin{array}{r} u = 1+2x \implies du = 2\,dx \\ dv = \mathrm{e}^{-x}\,dx \implies v = -\mathrm{e}^{-x} \end{array}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int_{-\infty}^{0} (1+2x)\,\mathrm{e}^{-x}\,dx = \lim_{R\to-\infty} -\mathrm{e}^{-x}(1+2x)\Big|_R^0 - \int_{-\infty}^{0} -\mathrm{e}^{-x}\cdot 2\,dx$$

$$= \lim_{R\to-\infty}\left(-\mathrm{e}^0\cdot 1 + e^{-R}(1+2\cdot R)\right) + \lim_{P\to-\infty} 2\mathrm{e}^{-x}\Big|_P^0$$

$$= -\infty + 2\lim_{P\to-\infty}\left(\mathrm{e}^0 - \mathrm{e}^{-P}\right) = -\infty - \infty = \boxed{-\infty}$$

The integral diverges to negative infinity.

7. The length of a curve defined by $x = f(y)$ whose derivative is continuous on the interval $a \le y \le b$ can be evaluated using the integral

$$S = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$$

Find $\dfrac{dx}{dy}$.

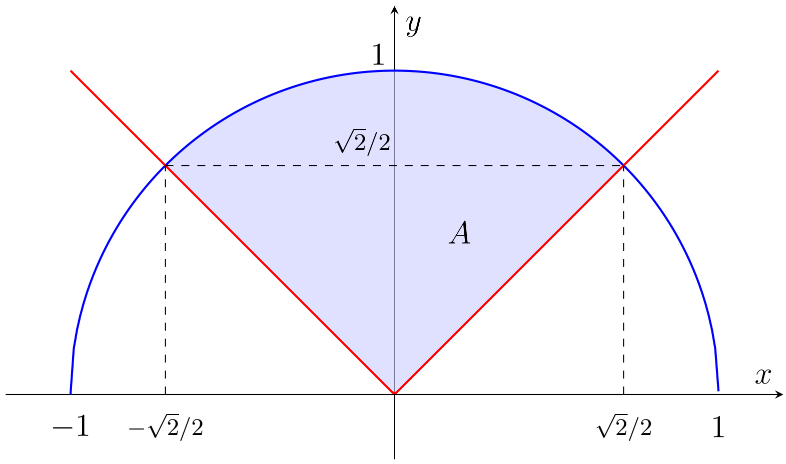
$$\frac{dx}{dy} = \frac{2}{3}\cdot\frac{3}{2}(y-1)^{1/2} = \sqrt{y-1}$$

Set $a = 1,\ b = 2$ and find the length.

$$S = \int_1^2 \sqrt{1 + \left(\sqrt{y-1}\right)^2}\,dy = \int_1^2 \sqrt{y}\,dy = \frac{2}{3}y^{3/2}\Big|_1^2 = \boxed{\frac{2}{3}(2\sqrt{2}-1)}$$

2015-2016 Final (12/01/2016) Solutions
(Last update: 30/08/2025 00:48)

1.



$$A = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left(\sqrt{1-x^2} - |x|\right)\,dx = \int_{-\sqrt{2}/2}^{0} \left(\sqrt{1-x^2} + x\right)\,dx + \int_{0}^{\sqrt{2}/2} \left(\sqrt{1-x^2} - x\right)\,dx$$

$$= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \sqrt{1-x^2}\,dx - \int_{0}^{\sqrt{2}/2} 2x\,dx \tag{33}$$

Calculate the right-hand integral (1).

$$\int_{0}^{\sqrt{2}/2} 2x\,dx = x^2\Big|_{0}^{\sqrt{2}/2} = \left(\frac{\sqrt{2}}{2}\right)^2 - (0)^2 = \frac{1}{2}$$

To calculate the left-hand integral in (1), we will use a trigonometric substitution. Let $x = \sin u$, then $dx = \cos u\,du$.

$$x = -\frac{\sqrt{2}}{2} \implies u = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}, \qquad x = \frac{\sqrt{2}}{2} \implies u = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

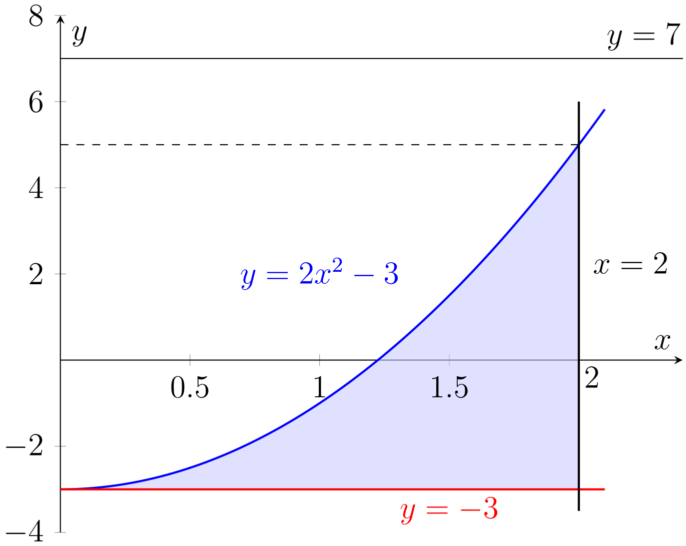
$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \sqrt{1-x^2}\,dx = \int_{-\pi/4}^{\pi/4} \sqrt{1-\sin^2 u}\cos u\,du = \int_{-\pi/4}^{\pi/4} |\cos u|\cos u\,du \quad [|\cos u| > 0]$$

$$= \int_{-\pi/4}^{\pi/4} \cos^2 u\,du = \int_{-\pi/4}^{\pi/4} \frac{1+\cos 2u}{2}\,du = \left[\frac{u}{2} + \frac{\sin 2u}{4}\right]_{-\pi/4}^{\pi/4}$$

$$= \left(\frac{\pi}{8} + \frac{1}{4}\right) - \left(-\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$$

Evaluate (1). The area is $A = \left(\frac{\pi}{4} + \frac{1}{2}\right) - \left(\frac{1}{2}\right) = \boxed{\frac{\pi}{4}}$.

2.

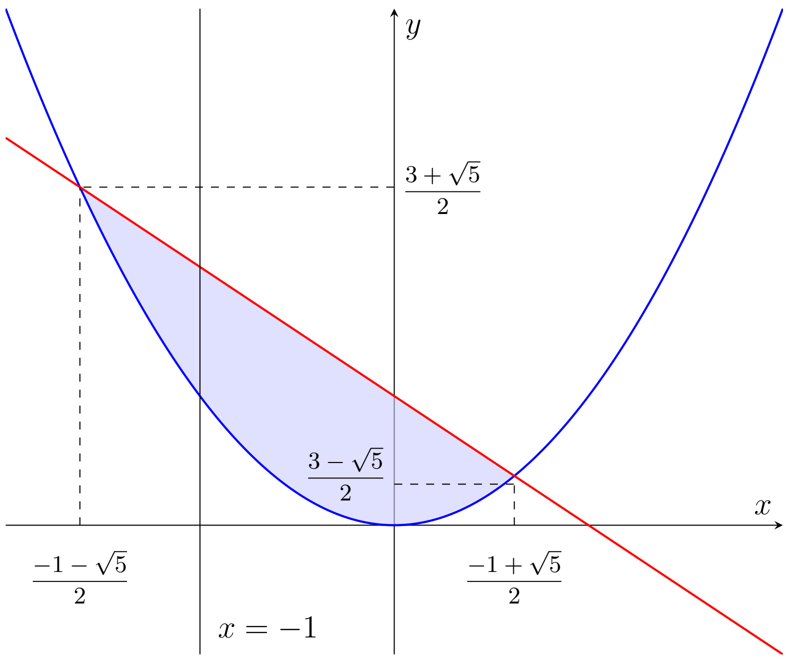


$$V=\int_D \pi\left[R_2^2(x)-R_1^2(x)\right]\,dx=\int_0^2 \pi\left[(7-(-3))^2-\left(7-\left(2x^2-3\right)\right)^2\right]\,dx$$

$$=\pi\int_0^2\left[(10)^2-\left(10-2x^2\right)^2\right]\,dx=\pi\int_0^2\left(100-100+40x^2-4x^4\right)\,dx$$

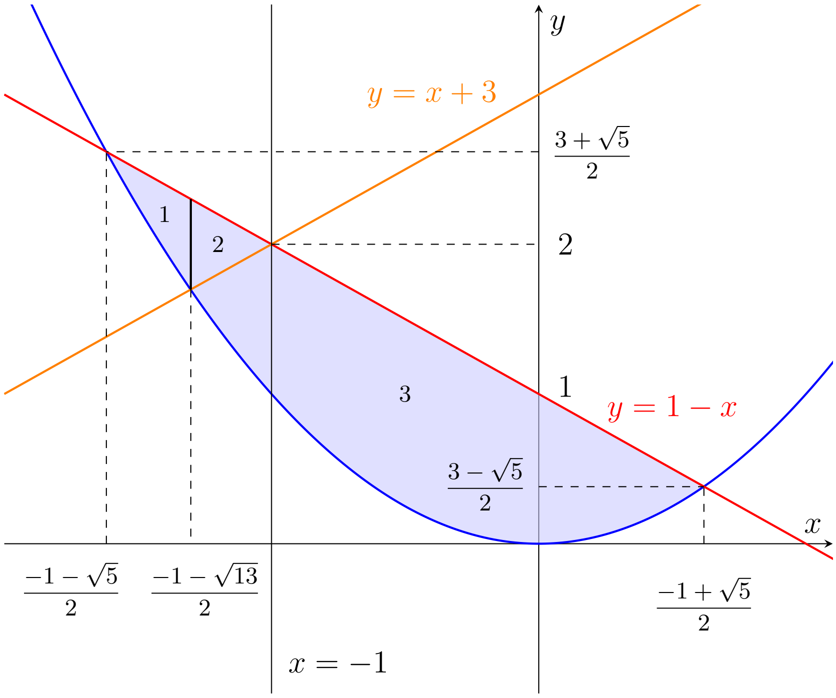
$$=\pi\int_0^2\left(40x^2-4x^4\right)\,dx=\pi\left[\frac{40x^3}{3}-\frac{4x^5}{5}\right]_0^2=\pi\left[\left(\frac{320}{3}-\frac{128}{5}\right)-(0)\right]=\boxed{\frac{1216\pi}{15}}$$

3.



Notice that the rotation axis passes through the region. Consider the right-hand region. If we rotate it around the axis, a piece of the region on the left will be inside the revolution. That is, the region bounded by the line that is symmetric to the line $y=-x+1$ around

$x=-1$ and the curve $y=x^2$. We do not need to rotate that region since it would lead to double revolution. The upper part of the left-hand region may be rotated around $x=-1$ independent of the right-hand region. Divide the left-hand region into three subregions. We get three different subregions to integrate over.



Let V be the volume of the solid.

$$V=\int_D 2\pi\cdot r(x)\cdot h(x)\,dx$$

$$=\int_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{13}}{2}} 2\pi(-1-x)\left[(1-x)-x^2\right]\,dx+\int_{\frac{1-\sqrt{13}}{2}}^{-1} 2\pi(-1-x)\left[(1-x)-(x+3)\right]\,dx$$

$$+\int_{-1}^{\frac{-1+\sqrt{5}}{2}} 2\pi\,(x+1)\left[(1-x)-x^2\right]\,dx$$

$$=2\pi\int_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{13}}{2}}\left(-1+2x^2+x^3\right)\,dx+2\pi\int_{\frac{1-\sqrt{13}}{2}}^{-1}\left(2x^2+4x+2\right)\,dx$$

$$+2\pi\int_{-1}^{\frac{-1+\sqrt{5}}{2}}\left(1-2x^2-x^3\right)\,dx$$

$$=2\pi\left\{\left[-x+\frac{2x^3}{3}+\frac{x^4}{4}\right]_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{13}}{2}}+\left[\frac{2x^3}{3}+2x^2+2x\right]_{\frac{1-\sqrt{13}}{2}}^{-1}+\left[x-\frac{2x^3}{3}-\frac{x^4}{4}\right]_{-1}^{\frac{-1+\sqrt{5}}{2}}\right\}$$

$$V = 2\pi\left[\frac{\sqrt{13}-1}{2} + \frac{\left(1-\sqrt{13}\right)^3}{12} + \frac{\left(1-\sqrt{13}\right)^4}{64} - \frac{1+\sqrt{5}}{2} + \frac{\left(1+\sqrt{5}\right)^3}{12} - \frac{\left(1+\sqrt{5}\right)^4}{64}\right]$$

$$+\,2\pi\left[-\frac{2}{3} + 2 - 2 - \left(\frac{\left(1-\sqrt{13}\right)^3}{12} + \frac{\left(1-\sqrt{13}\right)^2}{2} + 1 - \sqrt{13}\right)\right]$$

$$+\,2\pi\left[\frac{-1+\sqrt{5}}{2} - \frac{\left(-1+\sqrt{5}\right)^3}{12} - \frac{\left(-1+\sqrt{5}\right)^4}{64} - \left(-1 + \frac{2}{3} - \frac{1}{4}\right)\right]$$

After some mathematical operations, the volume is found to be

$$V = \frac{13\pi}{6} - \frac{\pi}{4}\left(47 - 13\sqrt{13}\right) = \boxed{\frac{\pi\left(39\sqrt{13} - 115\right)}{12}}$$

The volume can also be evaluated by integrating over the domain and subtracting the region that is revolved twice. That is, the region beneath the region 2 mapped on the graph.

4.

(a) Let $u = \sqrt{x}$, then $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin u \cdot 2\,du = -2\cos u + c = \boxed{-2\cos\sqrt{x} + c, \quad c \in \mathbb{R}}$$

(b) Perform a long polynomial division and rewrite the integral in two expressions.

$$\mathrm{I} = \int \frac{x^5 + x^4 - 8x^3 + 10x^2 + 12x}{x^2 - 3x + 2}\,dx = \int \left(x^3 - 4x^2 - 22x - 48 + \frac{-88x + 96}{x^2 - 3x + 2}\right)dx$$

$$= \int \left(x^3 + 4x^2 + 2x + 8\right)\,dx + \int \frac{32x - 16}{(x-2)(x-1)}\,dx \tag{34}$$

Calculate the integral on the left in (2).

$$\int \left(x^3 + 4x^2 + 2x + 8\right)\,dx = \frac{x^4}{4} + \frac{4x^3}{3} + x^2 + 8x + c_1$$

Calculate the integral on the right in (2). Decompose the expression into partial fractions.

$$\int \frac{32x - 16}{(x-2)(x-1)}\,dx = \int \left(\frac{A}{x-2} + \frac{B}{x-1}\right)\,dx$$

$$32x - 16 = A(x-1) + B(x-2) = x(A+B) - A - 2B$$

Equate the coefficients of like terms.

$$\left.\begin{array}{r} A+B=32 \\ -A-2B=-16 \end{array}\right\} \rightarrow A=48, \quad B=-16$$

Substitute the values into A and B.

$$\int \left(\frac{A}{x-2}+\frac{B}{x-1}\right) dx = \int \left(\frac{48}{x-2}-\frac{16}{x-1}\right) dx = 48\ln|x-2|-16\ln|x-1|+c_2$$

Rewrite (2).

$$\mathrm{I} = \boxed{\frac{x^4}{4}+\frac{4x^3}{3}+x^2+8x+48\ln|x-2|-16\ln|x-1|+c, \quad c\in\mathbb{R}}$$

(c) Apply integration by parts.

$$\left.\begin{array}{r} u=\arccos x \rightarrow du = -\dfrac{1}{\sqrt{1-x^2}}\,dx \\ dv=dx \rightarrow v=x \end{array}\right\} \rightarrow \int v\,du = uv - \int v\,du$$

$$\int \arccos x\,dx = x\arccos x - \int \frac{-x}{\sqrt{1-x^2}}\,dx$$

Let us use a u-substitution for the integral on the right. Let $u=1-x^2$, then $du=-2x\,dx$

$$\int \frac{-x}{\sqrt{1-x^2}}\,dx = \int \frac{du}{2\sqrt{u}} = \sqrt{u}+c = \sqrt{1-x^2}+c$$

Therefore,

$$\int \arccos x\,dx = \boxed{x\arccos x - \sqrt{1-x^2}+c, \quad c\in\mathbb{R}}$$

(d) Use the method of partial fraction decomposition.

$$\mathrm{I} = \int \frac{dx}{x^2+3x+1} = \int \frac{dx}{x^2+3x+\frac{9}{4}-\frac{5}{4}} = \int \frac{dx}{\left(x+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}$$

$$= \int \frac{dx}{\left(x+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)\left(x+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)} = \int \left(\frac{A}{x+\frac{3}{2}-\frac{\sqrt{5}}{2}}+\frac{B}{x+\frac{3}{2}+\frac{\sqrt{5}}{2}}\right) dx$$

$$A\left(x+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)+B\left(x+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)=1$$

$$x(A+B)+A\left(\frac{3+\sqrt{5}}{2}\right)+B\left(\frac{3-\sqrt{5}}{2}\right)=1$$

Equate the coefficients of like terms.

$$\left.\begin{aligned} A+B&=0\\ A\left(\tfrac{3+\sqrt{5}}{2}\right)+B\left(\tfrac{3-\sqrt{5}}{2}\right)&=1 \end{aligned}\right\} \implies A=\frac{1}{\sqrt{5}},\quad B=-\frac{1}{\sqrt{5}}$$

Substitute the values into A and B.

$$\mathrm{I}=\int\left(\frac{A}{x+\frac{3}{2}-\frac{\sqrt{5}}{2}}+\frac{B}{x+\frac{3}{2}+\frac{\sqrt{5}}{2}}\right)\,dx$$

$$=\int\left(\frac{1}{\sqrt{5}\left(x+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)}-\frac{1}{\sqrt{5}\left(x+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)}\right)\,dx$$

$$=\boxed{\frac{1}{\sqrt{5}}\left(\ln\left|x+\frac{3}{2}-\frac{\sqrt{5}}{2}\right|-\ln\left|x+\frac{3}{2}+\frac{\sqrt{5}}{2}\right|\right)+c,\quad c\in\mathbb{R}}$$

(e)

$$\mathrm{I}=\int\frac{\sin x}{1+\sin x}\,dx=\int\frac{1+\sin x-1}{1+\sin x}\,dx=\int dx-\int\frac{dx}{1+\sin x}$$

The integral on the left evaluates to $x+c_1$. Evaluate the other integral.

$$\int\frac{dx}{1+\sin x}=\int\frac{1-\sin x}{(1+\sin x)(1-\sin x)}\,dx=\int\frac{1-\sin x}{1-\sin^2 x}\,dx=\int\frac{1-\sin x}{\cos^2 x}\,dx$$

$$=\int\left(\sec^2 x-\tan x\sec x\right)\,dx=\tan x-\sec x+c_2$$

So, the result is

$$\mathrm{I}=\boxed{x-\tan x+\sec x+c,\quad c\in\mathbb{R}}$$

5. If the function $y=f(x)\geq 0$ is continuously differentiable on $[a,b]$, the area of the surface generated by revolving the graph of $y=f(x)$ about the x-axis is

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx}=\mathrm{e}^x$$

Set $a = 0$ and $b = 1$ and then evaluate the integral.

$$S = \int_0^1 2\pi \mathrm{e}^x \sqrt{1 + (\mathrm{e}^x)^2}\, dx$$

Let $u = \mathrm{e}^x$, then $du = \mathrm{e}^x\, dx$.

$$x = 0 \implies u = \mathrm{e}^0 = 1, \qquad x = 1 \implies u = \mathrm{e}^1 = \mathrm{e}$$

$$S = \int_0^1 2\pi \mathrm{e}^x \sqrt{1 + (\mathrm{e}^x)^2}\, dx = 2\pi \int_1^{\mathrm{e}} \sqrt{1 + u^2}\, du$$

We will now use a trigonometric substitution. Let $u = \tan t$ for $0 < t < \dfrac{\pi}{2}$, then $du = \sec^2 t\, dt$.

$$S = 2\pi \int_1^{\mathrm{e}} \sqrt{1 + u^2}\, du = 2\pi \int \sqrt{1 + \tan^2 t} \cdot \sec^2 t\, dt = 2\pi \int \sqrt{\sec^2 t} \cdot \sec^2 t\, dt$$

$$= 2\pi \int |\sec t| \sec^2 t\, dt = 2\pi \int \sec^3 t\, dt \qquad [\sec t > 0]$$

Find the antiderivative of $\sec^3 t$ with the help of integration by parts.

$$\begin{aligned} w = \sec t \ &\rightarrow \ dw = \sec t \tan t\, dt \\ dz = \sec^2 t\, dt \ &\rightarrow \ z = \tan t \end{aligned}$$

$$\int \sec^3 t\, du = \tan t \cdot \sec t - \int \tan^2 t \sec t\, dt = \tan t \cdot \sec t - \int \frac{1 - \cos^2 t}{\cos^3 t}\, dt$$

$$= \tan t \cdot \sec t - \int \sec^3 t\, dt + \int \sec t\, dt$$

Notice that the integral appears on the right side of the equation. Therefore,

$$\int \sec^3 t\, dt = \frac{1}{2} \cdot \tan t \cdot \sec t + \frac{1}{2} \cdot \int \sec t\, dt$$

The integral of $\sec t$ with respect to t is

$$\int \sec t\, dt = \ln|\tan t + \sec t| + c_1, \quad c_1 \in \mathbb{R}$$

Recall $u = \tan t$.

$$u = \tan t \implies u^2 = \tan^2 t = \sec^2 t - 1 \implies \sec t = \sqrt{u^2 + 1}$$

$$S = 2\pi \cdot \frac{1}{2}\left(\tan t \cdot \sec t + \ln\left|\tan t + \sec t\right|\right) + c = \pi\left[u \cdot \sqrt{u^2+1} + \ln\left|t + \sqrt{u^2+1}\right|\right]_1^{\mathrm{e}}$$

$$= \pi\left[\left(\mathrm{e} \cdot \sqrt{\mathrm{e}^2+1} + \ln\left|\mathrm{e} + \sqrt{\mathrm{e}^2+1}\right|\right) - \left(\sqrt{2} + \ln\left|1+\sqrt{2}\right|\right)\right]$$

$$= \boxed{\pi\left[\mathrm{e} \cdot \sqrt{\mathrm{e}^2+1} - \sqrt{2} + \ln\left(\frac{\mathrm{e} + \sqrt{\mathrm{e}^2+1}}{1+\sqrt{2}}\right)\right]}$$

6. Let $u = 1 + \mathrm{e}^{-x}$, then $du = -\mathrm{e}^{-x}\,dx$. Handle the improper integral by taking the limit.

$$x = 0 \implies u = 1 + \mathrm{e}^0 = 2, \qquad x \to -\infty \implies u \to \infty$$

$$\int_0^{-\infty} \frac{\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}}\,dx = \int_2^{\infty} -\frac{du}{u} = \lim_{R\to\infty}\int_2^R -\frac{du}{u} = \lim_{R\to\infty}\left(-\ln u\right)\Big|_2^R$$

$$= \lim_{R\to\infty}\left(-\ln R + \ln 2\right) = \boxed{-\infty}$$

The integral diverges to negative infinity.

7. According to the Monotone Convergence Theorem, if a sequence is both bounded and monotonic, the sequence converges. Take the corresponding function $f(x) = \dfrac{\ln x}{x}$. Apply the first derivative test and find the extrema.

$$f'(x) = \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1-\ln x}{x^2}$$

$$f'(x) = 0 \implies 1 - \ln x = 0 \implies \ln x = 1 \implies x = \mathrm{e}$$

A critical point occurs at $x = \mathrm{e}$. Apply the second derivative test and determine whether this is a local minimum or a local maximum.

$$f''(x) = \frac{-\frac{1}{x}\cdot x^2 - (1-\ln x)\cdot 2x}{x^4} = \frac{2\ln x - 3}{x^3}$$

$$f''(\mathrm{e}) = \frac{-1}{\mathrm{e}^3} < 0$$

Therefore, this is a local maximum.

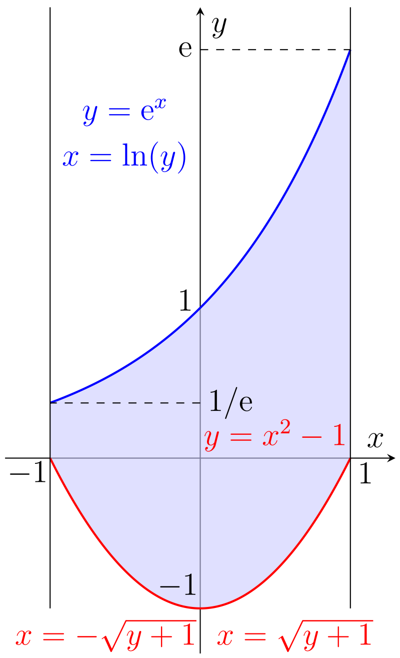
The first term of the sequence is $\dfrac{\ln 1}{1} = 0$. Take the limit at infinity. We may apply L'Hôpital's rule because we have an indeterminate form, which is ∞/∞.

$$\lim_{x\to\infty}\frac{\ln x}{x} \overset{\text{L'H.}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x} = 0$$

Since $\dfrac{\ln x}{x}$ is decreasing and bounded above by $f(\mathrm{e}) = \dfrac{1}{\mathrm{e}}$ and below by 0 for $x \geq \mathrm{e}$, by the Monotone Convergence Theorem, the sequence converges.

2017-2018 Summer Final (13/08/2018) Solutions
(Last update: 30/08/2025 01:00)

1.



$$\boxed{\begin{aligned} A = &\int_{-1}^{0} \left[\left(\sqrt{y+1}\right) - \left(-\sqrt{y+1}\right)\right] \, dy \\ &+ \int_{0}^{1/\mathrm{e}} [(1) - (-1)] \, dy + \int_{1/\mathrm{e}}^{\mathrm{e}} [(1) - (\ln y)] \, dy \end{aligned}}$$

2. The function f is defined where the denominator is not equal to zero. However, the denominator is always positive. Therefore, the domain of f is $\mathbb{R}$.

Let us find the limit at infinity and negative infinity.

$$\lim_{x\to\infty} \frac{4x}{x^2+1} = \lim_{x\to\infty} \frac{4}{x + \frac{1}{x}} = 0$$

Similarly, at negative infinity, the limit is zero. Therefore, the x-axis is a horizontal asymptote. There is no vertical asymptote.

Take the first derivative and find the critical points. Apply the quotient rule.

$$y' = \frac{d}{dx}\left(\frac{4x}{x^2+1}\right) = \frac{4 \cdot (x^2+1) - 4x \cdot (2x)}{(x^2+1)^2} = \frac{-4x^2+4}{(x^2+1)^2}$$

The critical points occur at $x = \pm 1$. At these points, the first derivative is 0.

Take the second derivative. Apply the quotient rule again.

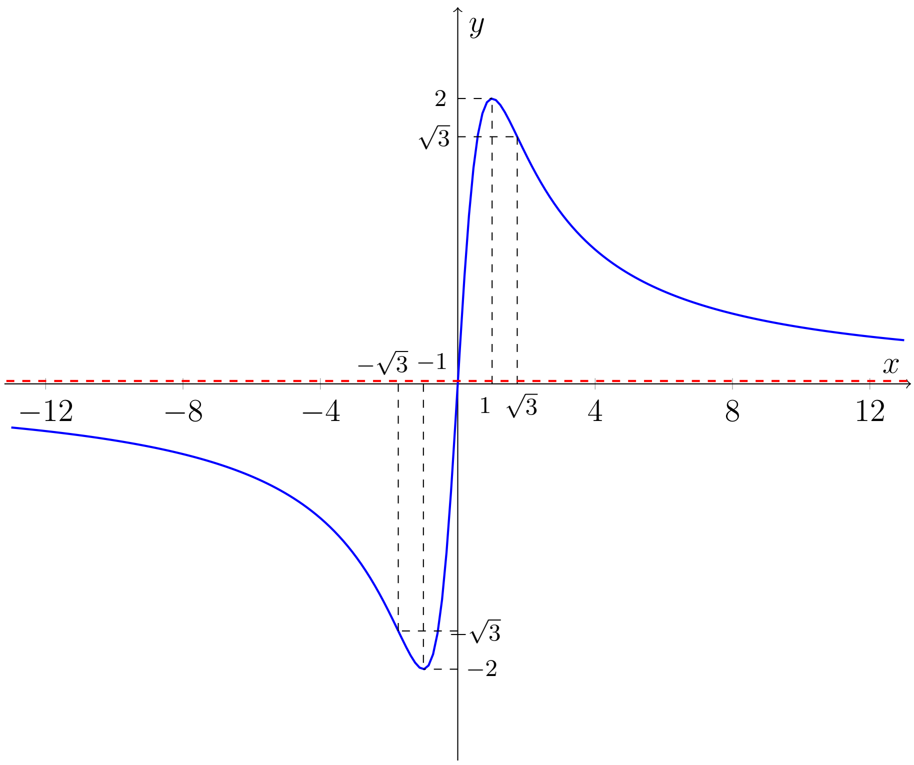
$$y'' = \frac{d}{dx}\left(\frac{-4x^2+4}{(x^2+1)^2}\right) = \frac{(-8x) \cdot (x^2+1)^2 - (-4x^2+4) \cdot 2(x^2+1) \cdot (2x)}{(x^2+1)^4} = \frac{8x(x^2-3)}{(x^2+1)^3}$$

The inflection points occur at $x = 0$ and $x = \pm\sqrt{3}$. At these points, the direction of the curvature changes.

Consider some values of the function. Eventually, set up a table and see what the graph looks like in certain intervals.

$$f\left(-\sqrt{3}\right)=-\sqrt{3},\quad f(-1)=-2,\quad f(0)=0,\quad f(1)=2,\quad f\left(\sqrt{3}\right)=\sqrt{3}$$

x	$\left(-\infty,-\sqrt{3}\right)$	$\left(-\sqrt{3},-1\right)$	$(-1,0)$	$(0,1)$	$\left(1,\sqrt{3}\right)$	$\left(\sqrt{3},\infty\right)$
y	$\left(-\sqrt{3},0\right)$	$\left(-2,-4\sqrt{3}\right)$	$(-2,0)$	$(0,2)$	$\left(\sqrt{3},2\right)$	$\left(0,\sqrt{3}\right)$
y' sign	-	-	+	+	-	-
y'' sign	-	+	+	-	-	+



3.

(a) Try to rewrite in the form of $\sin x$ and $\cos x$.

$$\int \frac{\tan x}{\sec^4 x}\,dx=\int \frac{\sin x}{\cos x}\cdot\cos^4 x\,dx=\int \sin x\cdot\cos^3 x\,dx=\int \sin x\cdot\left(1-\sin^2 x\right)\cdot\cos x\,dx$$

Let $u=\sin x$, then $du=\cos x\,dx$.

$$\int \sin x\cdot\left(1-\sin^2 x\right)\cdot\cos x\,dx=\int u\left(1-u^2\right)\,du=\int\left(u-u^3\right)\,du=\frac{u^2}{2}-\frac{u^4}{4}+c$$

$$=\boxed{\frac{\sin^2 x}{2}-\frac{\sin^4 x}{4}+c,\quad c\in\mathbb{R}}$$

(b) Use the method of integration by parts.

$$\left.\begin{array}{l} u = \ln x \implies du = \dfrac{1}{x}\,dx \\ dv = \dfrac{1}{x^3}\,dx \implies v = -\dfrac{1}{2x^2} \end{array}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int \frac{\ln x}{x^3}\,dx = -\frac{\ln x}{2x^2} - \int -\frac{1}{2x^3}\,dx = \boxed{-\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c, \quad c \in \mathbb{R}}$$

(c) Expand the expression by multiplying and dividing by the conjugate of the denominator.

$$\int \frac{1}{1+\sin x}\,dx = \int \frac{1-\sin x}{(1+\sin x)(1-\sin x)}\,dx = \int \frac{1-\sin x}{1-\sin^2 x}\,dx = \int \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \int \left(\sec^2 x - \tan x \sec x\right)\,dx = \boxed{\tan x - \sec x + c, \quad c \in \mathbb{R}}$$

(d) Use the method of partial fraction decomposition.

$$\int \frac{x+7}{x^2\,(x+2)}\,dx = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}\right)\,dx$$

$$\begin{aligned} A(x)(x+2) + B(x+2) + C\left(x^2\right) &= x+7 \\ x^2(A+C) + x(2A+B) + 2B &= x+7 \end{aligned}$$

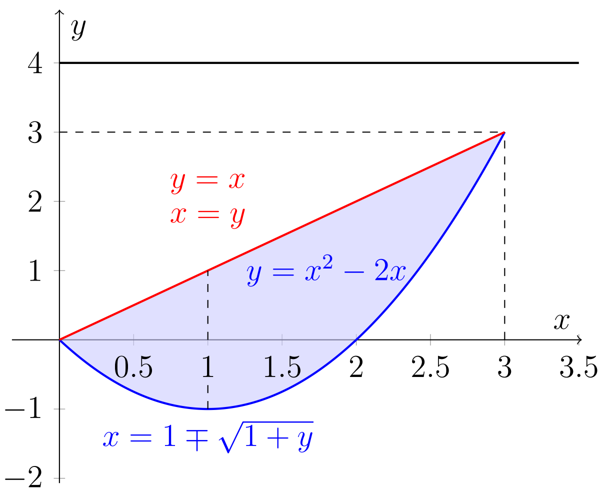
Equate the coefficients of like terms.

$$\left.\begin{array}{c} A + C = 0 \\ 2A + B = 1 \\ 2B = 7 \end{array}\right\} \implies A = -\frac{5}{4}, \quad B = \frac{7}{2}, \quad C = \frac{5}{4}$$

Rewrite the integral by substituting the values into the unknowns.

$$\int \left(-\frac{5}{4x} + \frac{7}{2x^2} + \frac{5}{4(x+2)}\right)\,dx = \boxed{-\frac{5}{4}\ln|x| - \frac{7}{2x} + \frac{5}{4}\ln|x+2| + c, \quad c \in \mathbb{R}}$$

4.



$$V = \int_D 2\pi \cdot h(y) \cdot r(y)\,dy$$

$$V = \int_{-1}^{0} 2\pi\,(4-y)\left[\left(1+\sqrt{1+y}\right) - \left(1-\sqrt{1+y}\right)\right]\,dy$$

$$+ \int_{0}^{3} 2\pi(4-y)\left[\left(1+\sqrt{1+y}\right) - (y)\right]\,dy$$

$$= 2\pi \int_{-1}^{0} (4-y)\left(2\sqrt{1+y}\right)\,dy + 2\pi \int_{0}^{3} (4-y)\left(1+\sqrt{1+y}-y\right)\,dy \tag{35}$$

Evaluate the first integral in (1). Let $u = 1+y$, then $du = dy$.

$$y = -1 \implies u = 0, \qquad y = 0 \implies u = 1$$

$$\mathrm{I}_1 = \int_{-1}^{0} (4-y)\left(2\sqrt{1+y}\right)\,dy = 2\int_{0}^{1} (5-u)\sqrt{u}\,du = 10\int_{0}^{1} \sqrt{u}\,du - 2\int_{0}^{1} u^{3/2}\,du$$

$$= \frac{20}{3}u^{3/2} - \frac{4}{5}u^{5/2}\Bigg|_{0}^{1} = \frac{20}{3} - \frac{4}{5} - 0 = \frac{88}{15}$$

Evaluate the second integral in (1). Let $u = 1+y$, then $du = dy$.

$$y = 0 \implies u = 1, \qquad y = 3 \implies u = 4$$

$$\mathrm{I}_2 = \int_{0}^{3} (4-y)\left(1+\sqrt{1+y}-y\right)\,dy = \int_{1}^{4} (5-u)\left(-u+2+\sqrt{u}\right)\,du$$

$$= \int_{1}^{4} \left(-7u+10+u^2+5\sqrt{u}-u^{3/2}\right)\,du = \left[-\frac{7u^2}{2} + 10u + \frac{u^3}{3} + \frac{10}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_{1}^{4}$$

$$= \left[\left(-56+40+\frac{64}{3}+\frac{80}{3}-\frac{64}{5}\right) - \left(-\frac{7}{2}+10+\frac{1}{3}+\frac{10}{3}-\frac{2}{5}\right)\right] = \frac{283}{30}$$

Therefore, the result is

$$2\pi\left(\frac{88}{15}+\frac{283}{30}\right) = \boxed{\frac{153\pi}{5}}$$

5. If the function $y = f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_{a}^{b} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$\dfrac{dy}{dx} = \mathrm{e}^x$. Set $a = 0,\ b = 1,$ and then evaluate the integral.

$$S=\int_0^1 2\pi \mathrm{e}^x\sqrt{1+(\mathrm{e}^x)^2}\,dx$$

Let $u=\mathrm{e}^x$, then $du=\mathrm{e}^x\,dx$.

$$x=0 \implies u=\mathrm{e}^0=1, \qquad x=1 \implies u=\mathrm{e}^1=\mathrm{e}$$

$$S=\int_0^1 2\pi \mathrm{e}^x\sqrt{1+(\mathrm{e}^x)^2}\,dx=2\pi\int_1^{\mathrm{e}}\sqrt{1+u^2}\,du$$

We will now use a trigonometric substitution. Let $u=\tan t$ for $0<t<\dfrac{\pi}{2}$, then $du=\sec^2 t\,dt$.

$$S=2\pi\int_1^{\mathrm{e}}\sqrt{1+u^2}\,du=2\pi\int\sqrt{1+\tan^2 t}\cdot\sec^2 t\,dt=2\pi\int\sqrt{\sec^2 t}\cdot\sec^2 t\,dt$$

$$=2\pi\int|\sec t|\sec^2 t\,dt=2\pi\int\sec^3 t\,dt \qquad [\sec t>0]$$

Find the antiderivative of $\sec^3 t$ with the help of integration by parts.

$$w=\sec t \to dw=\sec t\tan t\,dt$$
$$dz=\sec^2 t\,dt \to z=\tan t$$

$$\int\sec^3 t\,du=\tan t\cdot\sec t-\int\tan^2 t\sec t\,dt=\tan t\cdot\sec t-\int\frac{1-\cos^2 t}{\cos^3 t}\,dt$$

$$=\tan t\cdot\sec t-\int\sec^3 t\,dt+\int\sec t\,dt$$

Notice that the integral appears on the right side of the equation. Therefore,

$$\int\sec^3 t\,dt=\frac{1}{2}\cdot\tan t\cdot\sec t+\frac{1}{2}\cdot\int\sec t\,dt$$

The integral of $\sec t$ with respect to t is

$$\int\sec t\,dt=\ln|\tan t+\sec t|+c_1, \quad c_1\in\mathbb{R}$$

Recall $u=\tan t$.

$$u=\tan t \implies u^2=\tan^2 t=\sec^2 t-1 \implies \sec t=\sqrt{u^2+1}$$

$$S=2\pi\cdot\frac{1}{2}\left(\tan t\cdot\sec t+\ln|\tan t+\sec t|\right)+c=\pi\left[u\cdot\sqrt{u^2+1}+\ln\left|t+\sqrt{u^2+1}\right|\right]_1^{\mathrm{e}}$$

$$S = \pi\left[\left(\mathrm{e}\cdot\sqrt{\mathrm{e}^2+1}+\ln\left|\mathrm{e}+\sqrt{\mathrm{e}^2+1}\right|\right)-\left(\sqrt{2}+\ln\left|1+\sqrt{2}\right|\right)\right]$$

$$= \boxed{\pi\left[\mathrm{e}\cdot\sqrt{\mathrm{e}^2+1}-\sqrt{2}+\ln\left(\frac{\mathrm{e}+\sqrt{\mathrm{e}^2+1}}{1+\sqrt{2}}\right)\right]}$$

6. Let the corresponding function be $f(x) = x\mathrm{e}^{-x}$. f is continuous for $x \geq 0$. f is positive for $x \geq 0$ because $x \geq 0$ and e^{-x} is positive everywhere. The function is also decreasing for $x \geq 1$. Verify this behavior by taking the first derivative of f. Apply the product rule.

$$f'(x) = 1\cdot\mathrm{e}^{-x} - x\mathrm{e}^{-x} = (1-x)\mathrm{e}^{-x}$$

$f'(x) > 0$ for $x \geq 1$. The Integral Test states that all the conditions must be satisfied for and after a specific value, for instance $x = 1$. Therefore, set the lower bound $x = 1$ and evaluate the integral. We will exclusively evaluate the first term of the sequence thereafter.

$$\int_1^\infty x\mathrm{e}^{-x}\,dx$$

Apply integration by parts and then evaluate the improper integrals by taking the limit.

$$\left.\begin{aligned} u = x &\implies du = dv \\ dv = \mathrm{e}^{-x}\,dx &\implies v = -\mathrm{e}^{-x} \end{aligned}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int_1^\infty x\mathrm{e}^{-x}\,dx = \lim_{R\to\infty}\left[-x\mathrm{e}^{-x}\right]_1^R - \lim_{P\to\infty}\int_1^P -\mathrm{e}^{-x}\,dx = \lim_{R\to\infty}\left[-x\mathrm{e}^{-x}\right]_1^R - \lim_{P\to\infty}\left.\mathrm{e}^{-x}\right|_1^P$$

$$= \lim_{R\to\infty}\left(-R\mathrm{e}^{-R}+\mathrm{e}^{-1}\right) - \lim_{P\to\infty}\left(\mathrm{e}^{-P}-\mathrm{e}^{-1}\right) = \lim_{R\to\infty}\left(-R\mathrm{e}^{-R}\right) + 2\mathrm{e}^{-1}$$

To evaluate the limit, we assume that $-R\mathrm{e}^{-R}$ is a function of R. After that, put the expression in a form that we can apply L'Hôpital's rule in order to eliminate the form ∞/∞.

$$\lim_{R\to\infty}\left(-R\mathrm{e}^{-R}\right) = \lim_{R\to\infty}\frac{-R}{\frac{1}{\mathrm{e}^{-R}}} \overset{\text{L'H.}}{=} \lim_{R\to\infty}\frac{1}{\frac{1}{\mathrm{e}^{-2R}}\cdot\left(-\mathrm{e}^{-R}\right)} = \lim_{R\to\infty}\left(-\mathrm{e}^{-R}\right) = 0$$

Since the integral converges to $2\mathrm{e}^{-1}$, the series $\displaystyle\sum_{k=1}^{\infty} k\mathrm{e}^{-k}$ also converges. The first term of the series in the original question is $0\cdot\mathrm{e}^0 = 0$. The sum of a convergent series and a finite number is still finite. Therefore, the sum $\displaystyle\sum_{k=0}^{\infty} k\mathrm{e}^{-k}$ converges.

7. The Maclaurin series of f is as follows.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Find $f(0),\ f'(0),\ f''(0),\ f'''(0), f^{(4)}(0)$ to look for the pattern.

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3} \quad f^{(4)}(x) = -\frac{6}{(1+x)^4}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2, \quad f^{(4)}(0) = -6$$

This is an alternating sequence where the coefficient of each term is the factorial of the subsequent number starting from 0 except for $k = 0$, that is, the first term of the series. At $k = 0$, the first term is 0. So,

$$f^k(0) = \begin{cases} (-1)^{k-1} \cdot (k-1)!, & \text{if } k > 0 \\ 0, & \text{if } k = 0 \end{cases}$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0 + \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot (k-1)!}{k \cdot (k-1)!} x^k$$

$$= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots}$$

8. Apply the Ratio Test for absolute convergence, and apply other tests at the endpoints.

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1) \cdot (5)^{n+1}} \cdot \frac{n \cdot 5^n}{(x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-2) \cdot n}{(n+1) \cdot 5} \right| = \frac{|x-2|}{5} \cdot \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = \frac{|x-2|}{5}$$

$$\frac{|x-2|}{5} < 1 \implies |x-2| < 5 \implies -5 < x-2 < 5 \implies -3 < x < 7 \quad \text{(convergent)}$$

Now, take a look at the endpoints.

$$x = -3 \implies \sum_{n=1}^{\infty} \frac{(-5)^n}{n \cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This is an alternating series. The non-alternating part, which is $\frac{1}{n}$, is nonincreasing for $n \geq 1$ and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges. Try $x = 7$.

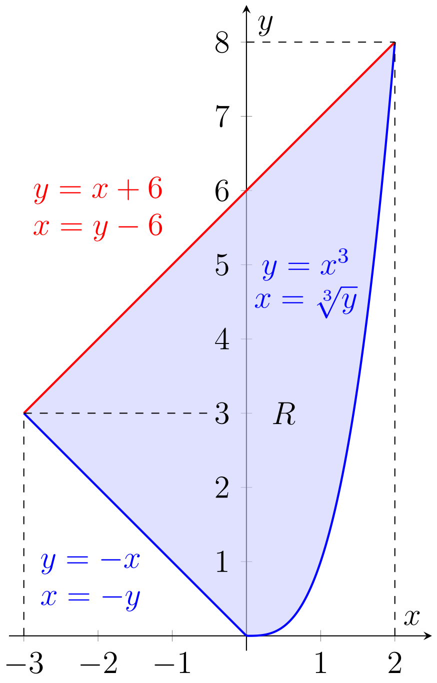
$$x = 7 \implies \sum_{n=1}^{\infty} \frac{(5)^n}{n \cdot 5^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

This is a p-series with $p = 1$, for which the series diverges by the p-series Test.

The convergence set for the power series is $\boxed{[-3, 7)}$.

2020-2021 Fall Final (18/01/2021) Solutions
(Last update: 16/08/2025 15:00)

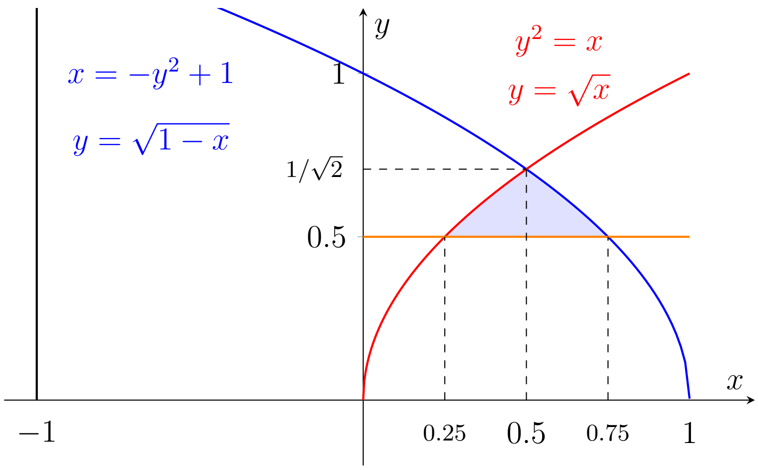
1.



(i)
$$\boxed{\begin{aligned}&\int_{-3}^{0} [(x+6) - (-x)]\ dx \\ &+ \int_{0}^{2} \left[(x+6) - \left(x^3\right)\right]\ dx\end{aligned}}$$

(ii)
$$\boxed{\begin{aligned}&\int_{0}^{3} [(\sqrt[3]{y}) - (-y)]\ dy \\ &+ \int_{3}^{8} [(\sqrt[3]{y}) - (y-6)]\ dy\end{aligned}}$$

2.



(i)
$$\boxed{\int_{1/4}^{1/2} 2\pi(x+1)\left[\left(\sqrt{x}\right) - \left(\frac{1}{2}\right)\right]\ dx + \int_{1/2}^{3/4} 2\pi(x+1)\left[\left(\sqrt{1-x}\right) - \left(\frac{1}{2}\right)\right]\ dx}$$

(ii)
$$\boxed{\int_{1/2}^{1/\sqrt{2}} \pi\left[\left(-y^2+1+1\right)^2 - \left(y^2+1\right)^2\right]\ dy}$$

3.

(a) Let $u = \sqrt[3]{x} + 4$, then $du = \dfrac{1}{3x^{2/3}}\, dx$.

$$\int \frac{dx}{x^{2/3}\left(\sqrt[3]{x}+4\right)} = \int \frac{3du}{u} = 3\ln|u| + c = \boxed{3\ln\left|\sqrt[3]{x}+4\right| + c, \quad c \in \mathbb{R}}$$

(b) Apply integration by parts.

$$\left.\begin{aligned} u = (\ln x)^2 &\implies du = 2\ln x \cdot \frac{1}{x}\, dx \\ dv = dx &\implies v = x \end{aligned}\right\} \to \int u\, dv = uv - \int v\, du$$

$$\int (\ln x)^2\, dx = x(\ln x)^2 - \int 2\ln x\, dx$$

Apply integration by parts once again.

$$\left.\begin{aligned} u = \ln x &\implies du = \frac{1}{x}\, dx \\ dv = dx &\implies v = x \end{aligned}\right\} \to \int u\, dv = uv - \int v\, du$$

$$\int (\ln x)^2\, dx = x(\ln x)^2 - \int 2\ln x\, dx$$

$$x(\ln x)^2 - \int 2\ln x\, dx = x(\ln x)^2 - 2\left[x\ln x - \int dx\right] = \boxed{x(\ln x)^2 - 2x\ln x + 2x + c,\ c \in \mathbb{R}}$$

(c) Let $x = \sqrt{3}\tan u$ for $0 < u < \dfrac{\pi}{2}$, then $dx = \sqrt{3}\sec^2 u\, du$.

$$\int \frac{dx}{\sqrt{3+x^2}} = \int \frac{\sqrt{3}\sec^2 u}{\sqrt{3+3\tan^2 u}}\, du = \int \frac{\sec^2 u}{\sqrt{1+\tan^2 u}}\, du = \int \frac{\sec^2 u}{|\sec u|}\, du$$

$$= \int \sec u\, du \qquad [\sec u > 0]$$

$$= \ln|\tan u + \sec u| + c, \quad c \in \mathbb{R}$$

Recall $x = \sqrt{3}\tan u$.

$$x^2 = 3\tan^2 u = 3\sec^2 u - 3 \implies \sec^2 u = \frac{x^2+3}{3} \implies \sec u = \frac{\sqrt{x^2+3}}{\sqrt{3}}$$

$$\ln|\tan u + \sec u| + c = \boxed{\ln\left|\frac{x}{\sqrt{3}} + \frac{\sqrt{x^2+3}}{\sqrt{3}}\right| + c, \quad c \in \mathbb{R}}$$

We can omit the constant part.

$$\boxed{\ln\left(x + \sqrt{x^2+3}\right) + c, \quad c \in \mathbb{R}}$$

(d) We may utilize the tangent half-angle substitution, which is also called the Weierstrass substitution. Let $t = \tan\left(\dfrac{x}{2}\right)$. After some mathematical operations, we get the following.

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2}\,dt$$

$$\int \frac{dx}{2+\sin x} = \int \frac{2}{1+t^2}\cdot\frac{1}{2+\frac{2t}{1+t^2}}\,dt = \int \frac{dt}{t^2+t+1} = \int \frac{dt}{t^2+t+\frac{1}{4}+\frac{3}{4}}$$

$$= \int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{4}{3}\int \frac{dt}{\frac{4}{3}\left(t+\frac{1}{2}\right)^2+1} = \frac{4}{3}\int \frac{dt}{\left(\frac{2}{\sqrt{3}}\right)^2\left(t+\frac{1}{2}\right)^2+1}$$

Let $u = \dfrac{2}{\sqrt{3}}\left(t+\dfrac{1}{2}\right)$, then $du = \dfrac{2}{\sqrt{3}}\,dt$.

$$\frac{2\sqrt{3}}{3}\int \frac{du}{u^2+1} = \frac{2\sqrt{3}}{3}\arctan u + c = \frac{2\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right)+c$$

$$= \boxed{\frac{2\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}\left(\tan\left(\frac{x}{2}\right)+\frac{1}{2}\right)\right)+c, \quad c\in\mathbb{R}}$$

4. Take $f(x) = \dfrac{x^2+1}{x^3}$. We have $f'(x) = -\dfrac{x^2+3}{x^4} < 0$ for all $x \geq 1$. That means f is decreasing for $x \geq 1$. We also have

$$f(1) = 2, \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{1}{x}+\frac{1}{x^3}\right) \qquad \Longrightarrow 0 < f(x) \leq 2, \quad x \geq 1$$

Since the sequence is bounded and monotonic, by the Monotone Convergence Theorem, the sequence converges.

5. Take the corresponding function $f(x) = \dfrac{1}{x^4+x^2}$. f is positive for $x \geq 1$ because $x^4 > 0$ and $x^2 > 0$. f is also continuous for $x \geq 1$ because the denominator is a polynomial whose *only* root is zero, which is out of the boundary of the integral. Investigate the monotonicity of f by taking the first derivative.

$$f'(x) = -\frac{4x^3+2x}{(x^4+x^2)^2} \implies f'(x) \leq 0 \text{ for } x \geq 1$$

We may now apply the Integral Test since the criteria have been satisfied.

$$\int_1^{\infty} \frac{dx}{x^4+x^2} = \int_1^{\infty} \frac{dx}{x^2\left(x^2+1\right)} = \int_1^{\infty} \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right) dx$$

$$= \lim_{R\to\infty} \int_1^{R} \left(\frac{1}{x^2} - \frac{1}{x^2+1}\right) dx = \lim_{R\to\infty} \left[-\frac{1}{x} - \arctan x\right]_1^R$$

$$= \lim_{R\to\infty} \left[\left(-\frac{1}{R} - \arctan R\right) - (-1 - \arctan 1)\right] = 1 - \frac{\pi}{4} \quad \text{(convergent)}$$

By the Integral Test, the series $\sum_{k=1}^{\infty} \frac{1}{k^4+k^2}$ also converges.

6. Apply the Ratio Test for absolute convergence, and apply other tests at the endpoints.

$$\lim_{n\to\infty} \left|\frac{(x+1)^{k+1}}{2(k+1)} \cdot \frac{2k}{(x+1)^k}\right| = \lim_{n\to\infty} \left|\frac{(x+1)\cdot k}{k+1}\right| = |x+1| \cdot \lim_{n\to\infty} \left|\frac{k}{k+1}\right| = |x+1|$$

$$|x+1| < 1 \implies -1 < x+1 < 1 \implies -2 < x < 0 \quad \text{(convergent)}$$

Now, take a look at the endpoints.

$$x = -2 \to \sum_{k=1}^{\infty} \frac{(-1)^k}{2k} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

This is an alternating series. The non-alternating part, which is $\frac{1}{k}$, is nonincreasing for $k \geq 1$ and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges. Try $x = 0$.

$$x = 0 \to \sum_{k=1}^{\infty} \frac{1^k}{2k} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k}$$

This is a p-series with $p = 1$, for which the series diverges by the p-series Test.

The convergence set for the power series is

$$\boxed{[-2, 0)}$$

7. The Maclaurin series of $\sin x$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

Rewrite the limit using this expansion.

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \left[\frac{1}{x} \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)\right] = \lim_{x\to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ...\right) = \boxed{1}$$

2021-2022 Final (07/01/2022) Solutions
(Last update: 16/08/2025 21:36)

1.

(a) Use the method of partial fraction decomposition.

$$\int \frac{dx}{x^3-4x^2+3x} = \int \frac{dx}{x(x-3)(x-1)} = \int \left(\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-1}\right) dx$$

$$\begin{aligned} A(x-3)(x-1) + Bx(x-1) + Cx(x-3) &= 1 \\ x^2(A+B+C) + x(-4A-B-3C) + 3A &= 1 \end{aligned}$$

Equate the coefficients of like terms.

$$\left.\begin{aligned} x^2(A+B+C) &= 0 \\ x(-4A-B-3C) &= 0 \\ 3A &= 1 \end{aligned}\right\} \to A = \frac{1}{3}, \qquad \left.\begin{aligned} B+C &= -\frac{1}{3} \\ B+3C &= -\frac{4}{3} \end{aligned}\right\} \to B = \frac{1}{6}, \quad C = -\frac{1}{2}$$

Rewrite the integral.

$$\int \left(\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-1}\right) dx = \int \left(\frac{1}{3x} + \frac{1}{6(x-3)} - \frac{1}{2(x-1)}\right) dx$$

$$= \boxed{\frac{1}{3}\ln|x| + \frac{1}{6}\ln|x-3| - \frac{1}{2}\ln|x-1| + c, \quad c \in \mathbb{R}}$$

(b) The limit is in the indeterminate form 0/0. Apply L'Hôpital's rule to eliminate the indeterminate form.

$$\lim_{x \to \frac{\pi}{2}} \frac{\displaystyle\int_{\pi/2}^{x} \ln(\sin t)\, dt}{\sin x - 1} \overset{\text{L'H.}}{=} \lim_{x \to \frac{\pi}{2}} \frac{\displaystyle\frac{d}{dx}\int_{\pi/2}^{x} \ln(\sin t)\, dt}{\cos x}$$

By the Fundamental Theorem of Calculus, we may rewrite the limit as follows.

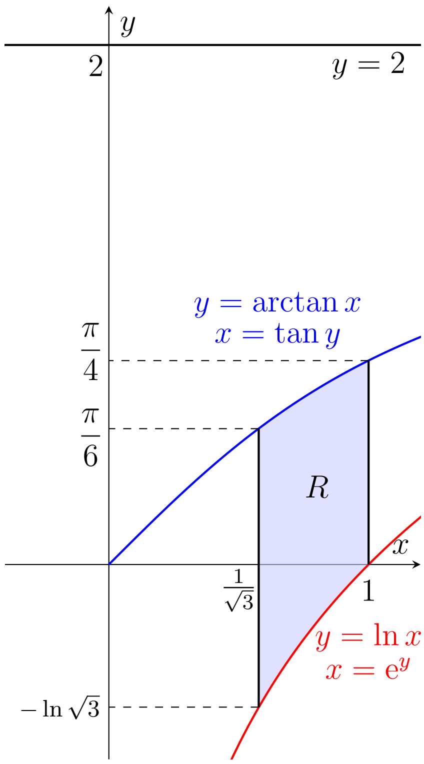
$$\lim_{x \to \frac{\pi}{2}} \frac{\displaystyle\frac{d}{dx}\int_{\pi/2}^{x} \ln(\sin t)\, dt}{\cos x} = \lim_{x \to \frac{\pi}{2}} \frac{\ln(\sin x)}{\cos x} \overset{\text{L'H.}}{=} \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\sin x}\cdot\cos x}{-\sin x} = \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{-\sin^2 x} = -\frac{\cos\frac{\pi}{2}}{\sin^2\frac{\pi}{2}}$$

$$= \boxed{0}$$

(c) Take the limit as this is an improper integral.

$$\int_1^2 \frac{dx}{(x-1)^{2/3}} = \lim_{R \to 1^+} \int_R^2 \frac{dx}{(x-1)^{2/3}} = \lim_{R \to 1^+} 3(x-1)^{1/3}\Big|_R^2 = 3\lim_{R \to 1^+}\left(1-(R-1)^{1/3}\right) = \boxed{3}$$

2.



(a)

$$A = \int_{1/\sqrt{3}}^{1} (\arctan x - \ln x)\ dx = \int_{1/\sqrt{3}}^{1} \arctan x\, dx - \int_{1/\sqrt{3}}^{1} \ln x\, dx \tag{36}$$

Calculate the first integral in (1) by integration by parts.

$$\left.\begin{array}{r} u = \arctan x \implies du = \dfrac{1}{x^2+1}\, dx \\ dv = dx \implies v = x \end{array}\right\} \to \int u\, dv = uv - \int v\, du$$

$$\int_{1/\sqrt{3}}^{1} \arctan x\, dx = x \arctan x \Big|_{1/\sqrt{3}}^{1} - \int_{1/\sqrt{3}}^{1} \frac{x}{x^2+1}\, dx = \left(x \arctan x - \frac{1}{2} \ln \left| x^2 + 1 \right| \right) \Bigg|_{1/\sqrt{3}}^{1}$$

$$= \left(\frac{\pi}{4} - \frac{\ln 2}{2} \right) - \left(\frac{\pi\sqrt{3}}{18} - \frac{1}{2} \cdot \ln \left(\frac{4}{3} \right) \right) = \frac{\pi \left(9 - 2\sqrt{3} \right)}{36} + \frac{1}{2} \cdot \ln \frac{2}{3}$$

Calculate the second integral in (1) by integration by parts.

$$\left.\begin{array}{r} u = \ln x \implies du = \dfrac{1}{x}\, dx \\ dv = dx \implies v = x \end{array}\right\} \to \int u\, dv = uv - \int v\, du$$

$$\int_{1/\sqrt{3}}^{1} \ln x\, dx = x \ln x\Big|_{1/\sqrt{3}}^{1} - \int_{1/\sqrt{3}}^{1} dx = (x \ln x - x)\Big|_{1/\sqrt{3}}^{1} = (0-1) - \left(-\frac{\ln\sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)$$

$$= \frac{\sqrt{3}\ln\sqrt{3} + \sqrt{3} - 3}{3}$$

The result is then

$$A = \boxed{\frac{\pi\left(9 - 2\sqrt{3}\right)}{36} + \frac{1}{2} \cdot \ln\frac{2}{3} - \frac{\sqrt{3}\ln\sqrt{3} + \sqrt{3} - 3}{3}}$$

(b)

$$\boxed{\begin{aligned}&\int_{-\ln\sqrt{3}}^{0} \pi\left[(e^y)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right] dy + \int_0^{\pi/6} \pi\left[(1)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right] dy \\ &+ \int_{\pi/6}^{\pi/4} \pi\left[(1^2) - (\tan y)^2\right] dy\end{aligned}}$$

(c)

$$\boxed{\begin{aligned}&\int_{-\ln\sqrt{3}}^{0} 2\pi\,(2-y)\left(e^y - \frac{1}{\sqrt{3}}\right) dy + \int_0^{\pi/6} 2\pi\,(2-y)\left(1 - \frac{1}{\sqrt{3}}\right) dy \\ &+ \int_{\pi/6}^{\pi/4} 2\pi\,(2-y)\,(1-\tan y)\; dy\end{aligned}}$$

3.

(a) Determine the nth partial sum.

$$\begin{aligned}\sum_{n=1}^{\infty} (\arctan n - \arctan(n-1)) &= (\cancel{\arctan 1} - \arctan 0) + (\cancel{\arctan 2} - \cancel{\arctan 1}) \\ &+ (\cancel{\arctan 3} - \cancel{\arctan 2}) + (\cancel{\arctan 4} - \cancel{\arctan 3}) + ... \\ &+ (\arctan n - \cancel{\arctan(n-1)}) = \arctan n - \arctan 0 \\ &= \arctan n\end{aligned}$$

$$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \quad \text{(convergent)}$$

Therefore, the series $\sum_{n=1}^{\infty} (\arctan n - \arctan(n-1))$ converges.

(b)

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}} = \sum_{n=0}^{\infty} \frac{-3 \cdot (-3)^n}{8^n} = -3 \sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n$$

This is a geometric series where $r = -\frac{3}{8}$. $|r| = \frac{3}{8} < 1$. Therefore, the series $\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$ converges.

(c) Apply the nth Term Test for divergence. We may take the limit inside the trigonometric function because $\cos x$ is continuous everywhere. Since $1 + n^4$ grows faster than n^2, the value of the expression $\frac{n^2}{1+n^4}$ tends to zero.

$$\lim_{n\to\infty} \cos\left(\frac{n^2}{1+n^4}\right) = \cos\left[\lim_{n\to\infty}\left(\frac{n^2}{1+n^4}\right)\right] = \cos 0 = 1 \neq 0$$

By the nth Term Test for divergence, the series $\sum_{n=1}^{\infty} \cos\left(\frac{n^2}{1+n^4}\right)$ diverges.

(d) Take $f(x) = \frac{1}{x(\ln x)^2}$. f is positive and decreasing for $x \geq 2$ because x and $(\ln x)^2$ are positive and increasing for $x \geq 2$. x is a polynomial which is defined everywhere and $(\ln x)^2$ is continuous for $x \geq 2$. Since we took into account every criterion, we may apply the Integral Test. Handle the improper integrals by taking the limit.

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{R\to\infty} \int_2^{R} \frac{dx}{x(\ln x)^2} = \lim_{R\to\infty} \left[-\frac{1}{\ln x}\right]_2^R = \lim_{R\to\infty}\left[-\frac{1}{\ln R} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$$

Since the integral converges, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ also converges.

2022-2023 Final (11/01/2023) Solutions
(Last update: 18/08/2025 01:51)

1.

(a) The volume of a sphere with radius r is

$$V = \frac{4}{3}\pi r^3$$

The differential of V is

$$dV = 4\pi r^2\, dr$$

The maximum error is known to be 1/8 inches. So, $|dr| \leq 1/8$. The maximum propagated error is then

$$dV = 4\pi \cdot 14^2 \cdot \frac{1}{8} = \boxed{98\pi \text{ in}^3.}$$

(b) $|dV| = 2$ at most. Solve the differential form for dr.

$$dr = \frac{dV}{4\pi r^2} = \frac{2}{4\pi \cdot 14^2} = \boxed{\frac{1}{392\pi} \text{ inches}}$$

2.

(a) Let $x = u^2$, then $u = \sqrt{x}$ and $dx = 2u\, du$.

$$\int \frac{dx}{\sqrt{x}\,(\sqrt{x}+2)} = \int \frac{2u}{u(u+2)}\, du = 2\int \frac{du}{u+2} = 2\ln|u+2| + c$$

$$= \boxed{2\ln\left|\sqrt{x}+2\right| + c,\ c \in \mathbb{R}}$$

(b) This question is beyond the scope of the curriculum, and students are not expected to solve it using the knowledge they have acquired in this course.

(c) Let $x = \sqrt{3}\sin u$ for $-\frac{\pi}{2} < u < \frac{\pi}{2}$, then $dx = \sqrt{3}\cos u\, du$.

$$\int \frac{dx}{\sqrt{3-x^2}} = \int \frac{\sqrt{3}\cos u}{\sqrt{3-3\sin^2 u}}\, du = \int \frac{\cos u}{\sqrt{\cos^2 u}}\, du = \int \frac{\cos u}{|\cos u|}\, du$$

$$= \int \frac{\cos u}{\cos u}\, du \quad [\cos u > 0]$$

$$= \int du = u + c$$

If $x = \sqrt{3}\sin u$, then $\sin u = \dfrac{x}{\sqrt{3}} \implies u = \arcsin\left(\dfrac{x}{\sqrt{3}}\right)$. The answer is then

$$\boxed{\arcsin\left(\frac{x}{\sqrt{3}}\right) + c, \quad c \in \mathbb{R}}$$

(d) We may utilize the tangent half-angle substitution, which is sometimes called the Weierstrass substitution. Let $t = \tan\left(\dfrac{x}{2}\right)$. Then

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2}\,dt$$

Rewrite the integral.

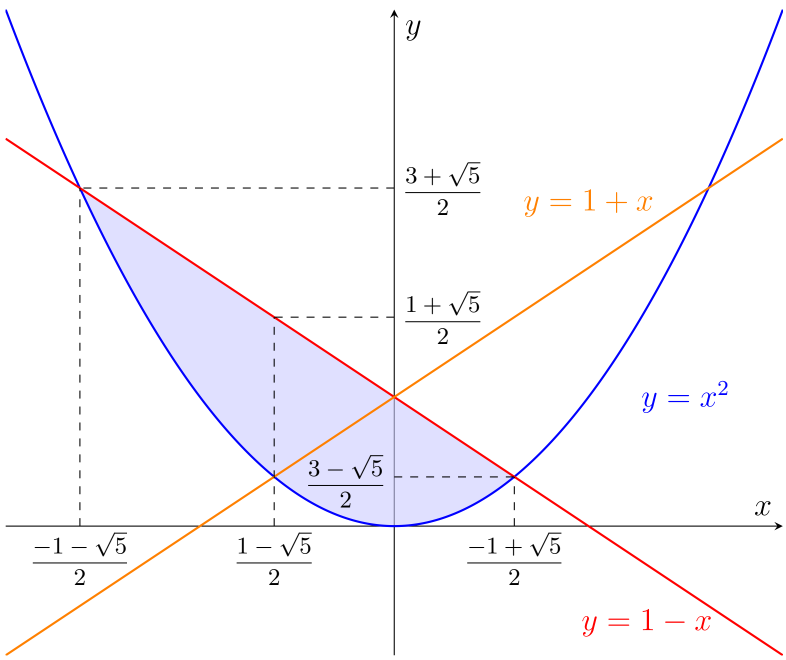
$$\int \frac{dx}{2+\cos x} = \int \frac{\frac{2}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}\,dt = \int \frac{2}{3+t^2}\,dt = \int \frac{2}{3\left(1+\frac{t^2}{3}\right)}\,dt = \frac{2}{3}\int \frac{1}{1+\left(\frac{t}{\sqrt{3}}\right)^2}\,dt$$

Let $u = \dfrac{t}{\sqrt{3}}$, then $\sqrt{3}\,du = dt$.

$$\frac{2}{3}\int \frac{dt}{1+\left(\frac{t}{\sqrt{3}}\right)^2} = \frac{2\sqrt{3}}{3}\int \frac{du}{1+u^2} = \frac{2\sqrt{3}}{3}\arctan u + c = \frac{2\sqrt{3}}{3}\arctan\frac{t}{\sqrt{3}} + c$$

$$= \boxed{\frac{2\sqrt{3}}{3}\arctan\left(\frac{1}{\sqrt{3}}\tan\left(\frac{x}{2}\right)\right) + c, \quad c \in \mathbb{R}}$$

3.



We have the symmetry of the region that is bounded to the right of the y-axis. Therefore, it is not necessary to apply the method to the symmetrical region on the left. The volume of this solid is

$$\boxed{\begin{aligned}&\int_{\frac{-1-\sqrt{5}}{2}}^{\frac{1-\sqrt{5}}{2}} 2\pi(-x)\left[(1-x)-\left(x^2\right)\right]\,dx\\&+\int_{\frac{1-\sqrt{5}}{2}}^{0} 2\pi(-x)\left[(1-x)-(1+x)\right]\,dx+\int_{0}^{\frac{-1+\sqrt{5}}{2}} 2\pi(x)\left[(1-x)-\left(x^2\right)\right]\,dx\end{aligned}}$$

4. If the function $y=f(x)\geq 0$ is continuously differentiable on $[a,b]$, the area of the surface generated by revolving the graph of $y=f(x)$ about the x-axis is

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$\dfrac{dy}{dx}=\dfrac{x^2}{2}-\dfrac{1}{2x^2}$. Set $a=1/2,\ b=1$ and then evaluate the integral.

$$S=\int_{1/2}^{1} 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\,dx$$

$$=\int_{1/2}^{1} 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{1+\frac{x^4}{4}-\frac{1}{2}+\frac{1}{4x^4}}\,dx=\int_{1/2}^{1} 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{\frac{x^4}{4}+\frac{1}{2}+\frac{1}{4x^4}}\,dx$$

$$=\int_{1/2}^{1} 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\sqrt{\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2}\,dx=\int_{1/2}^{1} 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)\,dx$$

$$=\int_{1/2}^{1} 2\pi\left(\frac{x^5}{12}+\frac{x}{12}+\frac{x}{4}+\frac{1}{4x^3}\right)\,dx=2\pi\left[\frac{x^6}{72}+\frac{x^2}{24}+\frac{x^2}{8}-\frac{1}{8x^2}\right]_{1/2}^{1}$$

$$=2\pi\left[\left(\frac{1}{72}+\frac{1}{24}+\frac{1}{8}-\frac{1}{8}\right)-\left(\frac{1}{4608}+\frac{1}{96}+\frac{1}{32}-\frac{1}{2}\right)\right]=\boxed{\frac{2367\pi}{2304}}$$

5. Take the corresponding function $f(x)=\dfrac{2}{3x+5}$. The function is continuous for $x\geq 1$ because the denominator is a first-degree polynomial whose root is $x_0=-\dfrac{5}{3}<1$. f is also positive and increasing for $x\geq 1$. Since the criteria hold, we may apply the Integral Test. Handle the improper integral by taking the limit.

$$\int_1^{\infty}\frac{2}{3x+5}\,dx=\lim_{R\to\infty}\int_1^R\frac{2}{3x+5}\,dx=\lim_{R\to\infty}\frac{2}{3}\ln|3x+5|\Big|_1^R=\frac{2}{3}\lim_{R\to\infty}\left(\ln|3R+5|-\ln 8\right)$$

$$=\infty$$

Since the integral diverges, by the Integral Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n+5}$ also diverges.

6. Recall the equality $\dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1}$.

$$f(x) = \frac{1}{x^2 - 5x + 6} = \frac{1}{(x-3)(x-2)} = \frac{1}{x-3} - \frac{1}{x-2}$$

The Maclaurin series of f is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, $f^{(4)}(0)$ to look for the pattern.

$$f'(x) = -\frac{1}{(x-3)^2} + \frac{1}{(x-2)^2}, \quad f''(x) = \frac{2}{(x-3)^3} - \frac{2}{(x-2)^3}$$
$$f'''(x) = -\frac{6}{(x-3)^4} + \frac{6}{(x-2)^4}, \quad f^{(4)}(x) = \frac{24}{(x-3)^5} - \frac{24}{(x-2)^5}$$

$$f(0) = -\frac{1}{3} + \frac{1}{2}, \quad f'(0) = -\frac{1}{9} + \frac{1}{4}, \quad f''(0) = -\frac{2}{27} + \frac{2}{8}$$
$$f'''(0) = -\frac{6}{81} + \frac{6}{16}, \quad f^{(4)}(0) = -\frac{24}{243} + \frac{24}{32}$$

This is a sequence where each term is defined by the following.

$$f^{(k)}(0) = (k!) \cdot \left(-\frac{1}{3^{k+1}} + \frac{1}{2^{k+1}} \right)$$

Rewrite the sum.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{(k!) \cdot x^k}{k!} \cdot \left(-\frac{1}{3^{k+1}} + \frac{1}{2^{k+1}} \right)$$

$$= \boxed{\sum_{k=0}^{\infty} x^k \left(\frac{1}{2^{k+1}} - \frac{1}{3^{k+1}} \right)}$$

2022-2023 Final (13/01/2023) Solutions
(Last update: 20/08/2025 16:38)

1.

(a) Let $x = 2\sin u$ for $-\dfrac{\pi}{2} < u < \dfrac{\pi}{2}$, then $dx = 2\cos u\, du$.

$$x = 0 \implies 2\sin u = 0 \implies u = 0$$

$$x = 1 \implies 2\sin u = 1 \implies \sin u = \frac{1}{2} \implies u = \arcsin\frac{1}{2} = \frac{\pi}{6}$$

$$\mathrm{I} = \int_0^1 \frac{x^2}{\sqrt{4-x^2}}\, dx = \int_0^{\pi/6} \frac{4\sin^2 u}{\sqrt{4-4\sin^2 u}} \cdot 2\cos u\, du \int_0^{\pi/6} \frac{4\sin^2 u\cos u}{|\cos u|}\, du \quad [\cos u > 0]$$

$$= \int_0^{\pi/6} 4\sin^2 u\, du = 4\int_0^{\pi/6} \left(1-\cos^2 u\right)\, du = 4\int_0^{\pi/6} \frac{1-\cos 2u}{2}\, du$$

$$= 4\left(\frac{u}{2} - \frac{\sin 2u}{4}\right)\Bigg|_0^{\pi/6} = 2u - \sin 2u\Bigg|_0^{\pi/6} = \left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) - 0 = \boxed{\frac{\pi}{3} - \frac{\sqrt{3}}{2}}$$

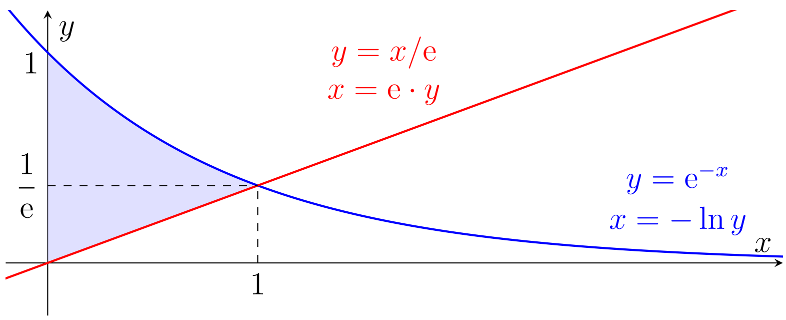
(b) Use the method of integration by parts.

$$\left.\begin{aligned} u = x &\implies du = dx \\ dv = \sec^2 x\, dx &\implies v = \tan x \end{aligned}\right\} \to \int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

$$\int_0^{\pi/4} x\sec^2 x\, dx = x\tan x\Big|_0^{\pi/4} - \int_0^{\pi/4} \tan x\, dx = x\tan x + \ln|\cos x|\Big|_0^{\pi/4} = \boxed{\frac{\pi}{4} + \ln\frac{\sqrt{2}}{2}}$$

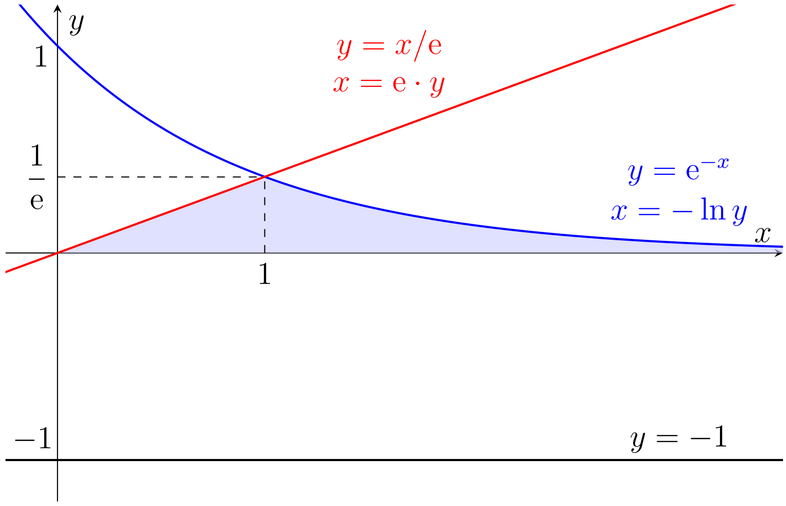
2.

(a)



$$V = \int_D 2\pi \cdot h(y) \cdot r(y)\, dy = \boxed{\int_0^{1/\mathrm{e}} 2\pi \cdot y \cdot (\mathrm{e}\cdot y - 0)\, dy + \int_{1/\mathrm{e}}^1 2\pi \cdot y \cdot (-\ln y - 0)\, dy}$$

(b)



$$V = \int_D \pi \left[r_2^2(x) - r_1^2(x) \right] \, dx = \boxed{\int_0^1 \left[\left(\frac{x}{\mathrm{e}} + 1 \right)^2 - 1^2 \right] \, dx + \int_1^{\infty} \left[\left(\mathrm{e}^{-x} + 1 \right)^2 - 1^2 \right] \, dx}$$

(c)

$$V = 2\pi \left[\int_0^{1/\mathrm{e}} \mathrm{e} y^2 \, dy - \int_{1/\mathrm{e}}^{1} y \ln y \, dy \right] \tag{37}$$

Calculate the integral on the left in (1).

$$\int_0^{1/\mathrm{e}} \mathrm{e} y^2 \, dy = \left. \frac{\mathrm{e} y^3}{3} \right|_0^{1/\mathrm{e}} = \frac{1}{3\mathrm{e}^2}$$

Calculate the integral on the right in (1) by using the method of integration by parts.

$$\left. \begin{aligned} u = \ln x \implies du &= \frac{1}{x} \, dx \\ dv = x \, dx \implies v &= \frac{x^2}{2} \end{aligned} \right\} \to \int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$$

$$\int_{1/\mathrm{e}}^{1} x \ln x \, dx = \left. \frac{x^2 \ln x}{2} \right|_{1/\mathrm{e}}^{1} - \int_{1/\mathrm{e}}^{1} \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \left. \frac{x^2 \ln x}{2} - \frac{x^2}{4} \right|_{1/\mathrm{e}}^{1}$$

$$= \left(0 - \frac{1}{4} \right) - \left(-\frac{1}{2\mathrm{e}^2} - \frac{1}{4\mathrm{e}^2} \right) = \frac{3}{4\mathrm{e}^2} - \frac{1}{4}$$

Therefore,

$$V = 2\pi \left[\left(\frac{1}{3\mathrm{e}^2} \right) - \left(\frac{3}{4\mathrm{e}^2} - \frac{1}{4} \right) \right] = \boxed{\pi \left(\frac{1}{2} - \frac{5}{6\mathrm{e}^2} \right)}$$

3.

(a) Apply the nth Term Test for divergence. The inverse trigonometric function arctan is continuous on $\mathbb{R}$. Therefore, we may take the limit inside the function.

$$\lim_{n\to\infty} \frac{1}{\arctan(n^2)} = \frac{1}{\arctan\left(\lim\limits_{n\to\infty} n^2\right)} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \neq 1$$

By the nth Term Test for divergence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\arctan(n^2)}$ diverges.

(b) Recall the sine inequality $-\theta \leq \sin\theta \leq \theta$. Then for all $n \in \mathbb{R}$ except zero, we have $-\dfrac{1}{n^3} \leq \sin\left(\dfrac{1}{n^3}\right) \leq \dfrac{1}{n^3}$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges because it is a p-series with $p = 3 > 1$. By the p-series Test, the series converges. The series $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^3}\right)$ also converges by the Direct Comparison Test because $\sin\left(\dfrac{1}{n^3}\right) < \dfrac{1}{n^3}$ for every $n \geq 1$.

(c) Take $f(x) = x\mathrm{e}^{-x^2}$. f is continuous because the product of a polynomial and an exponential expression is still continuous. f is positive and decreasing for $x \geq 1$. Verify the monotonicity of f by taking the first derivative.

$$\frac{df}{dx} = 1 \cdot \mathrm{e}^{-x^2} + x\mathrm{e}^{-x^2} \cdot (-2x) = \mathrm{e}^{-x^2}\left(1 - 2x^2\right)$$

$$f'(x) < 0 \quad \text{for} \quad x > \frac{\sqrt{2}}{2} \implies f'(x) < 0 \quad \text{for} \quad x \geq 1$$

We may now apply the Integral Test. Take the limit for the improper integral.

$$\int_1^{\infty} x\mathrm{e}^{-x^2}\, dx = \lim_{R\to\infty} \int_1^{R} x\mathrm{e}^{-x^2}\, dx = \lim_{R\to\infty} -\frac{1}{2}\mathrm{e}^{-x^2}\Big|_1^R = \lim_{R\to\infty} -\frac{1}{2}\left(\mathrm{e}^{-R^2} - \mathrm{e}^{-1}\right) = \frac{1}{2\mathrm{e}}$$

The integral converges. Then the series $\displaystyle\sum_{n=1}^{\infty} n\mathrm{e}^{-n^2}$ also converges.

4.

(a) Apply the Ratio Test for absolute convergence, and apply other tests at the endpoints.

$$\lim_{n\to\infty} \left| \frac{2^{n+2} \cdot (x+1)^{n+1}}{(n+1) \cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{(x+1)^n \cdot 2^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2n \cdot (x+1)}{(n+1) \cdot 3} \right|$$

$$= \frac{2|x+1|}{3} \cdot \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = \frac{2|x+1|}{3}$$

$$\frac{2|x+1|}{3} < 1 \implies |x+1| < \frac{3}{2}$$

The radius of convergence is $\boxed{\frac{3}{2}}$.

$$|x+1| < \frac{3}{2} \implies -\frac{3}{2} < x+1 < \frac{3}{2} \implies -\frac{5}{2} < x < \frac{1}{2} \quad \text{(convergent)}$$

Investigate the convergence at the endpoints.

$$x = \frac{1}{2} \implies \sum_{n=1}^{\infty} \frac{2^{n+1} \cdot \left(\frac{3}{2}\right)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{2}{n} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n}$$

This is a p-series with $p = 1$, for which the series diverges by the p-series Test. Try the other endpoint.

$$x = -\frac{5}{2} \implies \sum_{n=1}^{\infty} \frac{2^{n+1} \cdot \left(-\frac{3}{2}\right)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{2 \cdot (-1)^n}{n} = 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This is an alternating series. The non-alternating part, which is $\frac{1}{n}$, is nonincreasing for $n \geq 1$ and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges.

The convergence set for the power series is $\boxed{\left[-\frac{5}{2}, \frac{1}{2}\right)}$.

(b)

$$f(x) = \frac{x^{123}}{1+x^4} = x^{123} \cdot \frac{1}{1-(-x^4)} = x^{123} \cdot \sum_{n=0}^{\infty} \left(-x^4\right) = x^{123} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot x^{4n}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \cdot x^{123+4n} = x^{123} - x^{127} + x^{131} - x^{135} + ...}$$

2024-2025 Final (09/01/2025) Solutions
(Last update: 21/08/2025 00:12)

1.

(a) Use the tangent half-angle substitution, which is also called the Weierstrass substitution. Let $t = \tan\left(\frac{x}{2}\right)$. Then

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2}\,dt$$

$$\int \frac{dx}{2\cos x + 3} = \int \frac{1}{2\left(\frac{1-t^2}{1+t^2}\right)+3} \cdot \frac{2}{1+t^2}\,dt = \int \frac{2}{5+t^2}\,dt = \frac{2}{5}\int \frac{1}{\left(1+\left(\frac{t}{\sqrt{5}}\right)^2\right)}\,dt$$

Let $u = \frac{t}{\sqrt{5}}$, then $\sqrt{5}\,du = dt$.

$$\frac{2}{5}\int \frac{1}{\left(1+\left(\frac{t}{\sqrt{5}}\right)^2\right)}\,dt = \frac{2\sqrt{5}}{5}\int \frac{1}{1+u^2}\,du = \frac{2\sqrt{5}}{5}\arctan u + c = \frac{2\sqrt{5}}{5}\arctan \frac{t}{\sqrt{5}} + c$$

$$= \boxed{\frac{2\sqrt{5}}{5}\arctan\left(\frac{1}{\sqrt{5}} \cdot \tan\left(\frac{x}{2}\right)\right) + c, \quad c \in \mathbb{R}}$$

(b) Let $x = \sqrt{5}\sin u$ for $-\frac{\pi}{2} \le u < \frac{\pi}{2}$, then $dx = \sqrt{5}\cos u\,du$.

$$\mathrm{I} = \int \frac{\sqrt{5-x^2}}{x}\,dx = \int \frac{\sqrt{5-5\sin^2 u}}{\sqrt{5}\sin u} \cdot \sqrt{5}\cos u\,du \qquad \left[\sin^2 u + \cos^2 u = 1\right]$$

$$= \sqrt{5}\int \sqrt{\cos^2 u}\cot u\,du \qquad [\cos u > 0]$$

$$= \sqrt{5}\int \cos u \cdot \cot u\,du = \sqrt{5}\int \frac{\cos^2 u}{\sin u}\,du = \sqrt{5}\int \frac{1-\sin^2 u}{\sin u}\,du$$

$$= \sqrt{5}\int (\csc u - \sin u)\ du = \sqrt{5}\left(-\ln|\cot u + \csc u| + \cos u\right) + c$$

Recall that $x = \sqrt{5}\sin u$.

$$\sin u = \frac{x}{\sqrt{5}} \implies \sin^2 u = \frac{x^2}{5} \implies \cos^2 u = \frac{5-x^2}{5} \implies \cos u = \frac{\sqrt{5-x^2}}{\sqrt{5}}$$

$$\cot u = \frac{\cos u}{\sin u} = \frac{\sqrt{5-x^2}}{x}, \qquad \csc u = \frac{1}{\sin u} = \frac{\sqrt{5}}{x}$$

Therefore,

$$\mathrm{I} = \boxed{-\sqrt{5}\ln\left|\frac{\sqrt{5-x^2}}{x} + \frac{\sqrt{5}}{x}\right| + \sqrt{5-x^2} + c, \quad c \in \mathbb{R}}$$

(c) Use partial fractions to compute the integral.

$$\mathrm{I} = \int \frac{1}{1-x^4}\,dx = \int \frac{1}{(1-x^2)(1+x^2)}\,dx = \int \frac{1}{(1-x)(1+x)(1+x^2)}\,dx$$

$$= \int \left(\frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}\right)dx$$

$$\begin{gathered} A(1+x)(1+x^2) + B(1-x)(1+x^2) + (Cx+D)(1+x)(1-x) = 1 \\ x^3(A-B-C) + x^2(A+B-D) + x(A-B+C) + A+B+D = 1 \end{gathered}$$

Equate the coefficients of like terms.

$$\left.\begin{aligned} A-B-C&=0 \quad (1)\\ A+B-D&=0 \quad (2)\\ A-B+C&=0 \quad (3)\\ A+B+D&=1 \quad (4)\end{aligned}\right\} \qquad \begin{gathered} (1)\ \&\ (3) \to 2A-2B=0 \\ (2)\ \&\ (4) \to 2A+2B=1 \\ \therefore A=\frac{1}{4}, \quad B=\frac{1}{4} \\ (1)\to C=0, \quad (2)\to D=\frac{1}{2}\end{gathered}$$

Rewrite the integral.

$$\mathrm{I} = \int \left(\frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}\right)dx = \int \left(\frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)}\right)dx$$

$$= \boxed{-\frac{1}{4}\ln|1-x| + \frac{1}{4}\ln|1+x| + \frac{1}{2}\arctan(x) + c,\ c\in\mathbb{R}}$$

(d) Use the method of integration by parts.

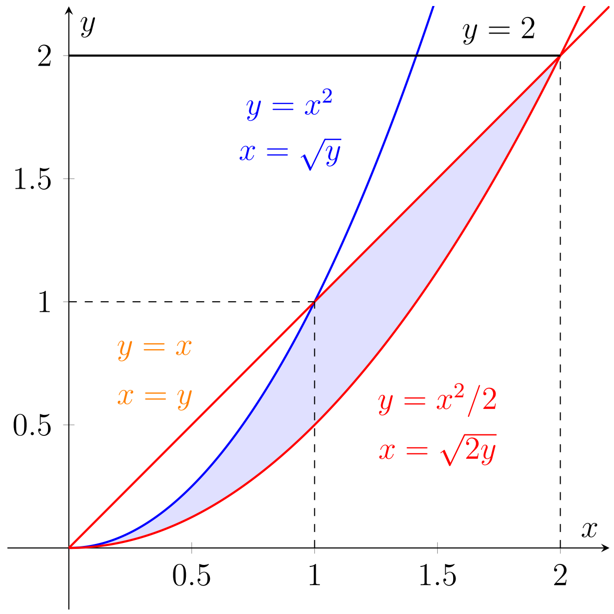
$$\left.\begin{aligned} u = \ln(1+x^2) &\implies du = \frac{2x}{1+x^2}\,dx \\ dv = dx &\implies v = x \end{aligned}\right\} \to \int u\,dv = uv - \int v\,du$$

$$\int \ln\left(1+x^2\right)dx = x\ln\left(1+x^2\right) - \int \frac{2x^2}{1+x^2}\,dx = x\ln\left(1+x^2\right) - \int \frac{2x^2+2-2}{1+x^2}\,dx$$

$$= x\ln\left(1+x^2\right) - 2\int dx + 2\int \frac{dx}{1+x^2}$$

$$= \boxed{x\ln\left(1+x^2\right) - 2x + 2\arctan x + c, \quad c\in\mathbb{R}}$$

2.



(a)

$$A = \int_0^1 \left(x^2 - \frac{x^2}{2}\right)\, dx + \int_1^2 \left(x - \frac{x^2}{2}\right)\, dx = \left[\frac{x^3}{3} - \frac{x^3}{6}\right]_0^1 + \left[\frac{x^2}{2} - \frac{x^3}{6}\right]_1^2$$

$$= \left[\left(\frac{1}{3} - \frac{1}{6}\right) - 0\right] + \left[\left(2 - \frac{8}{6}\right) - \left(\frac{1}{2} - \frac{1}{6}\right)\right] = \boxed{\frac{1}{2}}$$

(b)

$$V = \int_D \pi\left[r_2^2(y) - r_1^2(y)\right]\, dy = \boxed{\int_0^1 \pi\left[\left(\sqrt{2y}\right)^2 - (\sqrt{y})^2\right]\, dy + \int_1^2 \pi\left[\left(\sqrt{2y}\right)^2 - y^2\right]\, dy}$$

(c)

$$V = \int_D 2\pi \cdot h(y) \cdot r(y)\, dy = \boxed{\int_0^1 2\pi(2-y)\left(\sqrt{2y} - \sqrt{y}\right) dy + \int_1^2 2\pi(2-y)\left(\sqrt{2y} - y\right) dy}$$

3. Take $f(x) = x\mathrm{e}^{-x^2}$. f is continuous because the product of a polynomial and an exponential expression is still continuous. f is positive and decreasing for $x \geq 1$. Verify the monotonicity of f by taking the first derivative.

$$\frac{df}{dx} = 1 \cdot \mathrm{e}^{-x^2} + x\mathrm{e}^{-x^2} \cdot (-2x) = \mathrm{e}^{-x^2}\left(1 - 2x^2\right)$$

$$f'(x) < 0 \quad \text{for} \quad x > \frac{\sqrt{2}}{2} \implies f'(x) < 0 \quad \text{for} \quad x \geq 1$$

We may now apply the Integral Test. Take the limit for the improper integral.

$$\int_1^\infty x\mathrm{e}^{-x^2}\, dx = \lim_{R\to\infty} \int_1^R x\mathrm{e}^{-x^2}\, dx = \lim_{R\to\infty} -\frac{1}{2}\mathrm{e}^{-x^2}\bigg|_1^R = \lim_{R\to\infty} -\frac{1}{2}\left(\mathrm{e}^{-R^2} - \mathrm{e}^{-1}\right) = \frac{1}{2\mathrm{e}}$$

The integral converges. Then the series $\sum_{n=1}^{\infty} ne^{-n^2}$ also converges.

4. Apply the nth Term Test for the non-alternating part. Let L be the value of the limit.

$$L = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} \implies \ln(L) = \ln\left[\lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/n}\right]$$

We can also take the limit inside the function because ln is continuous on its domain.

$$\ln(L) = \lim_{n\to\infty} \ln\left[\left(\frac{1}{n}\right)^{1/n}\right] = \lim_{n\to\infty} \frac{\ln\left(\frac{1}{n}\right)}{n}$$

To be able to use L'Hôpital's rule, take the corresponding function $f(x) = \dfrac{\ln\left(\frac{1}{x}\right)}{x}$.

$$\lim_{x\to\infty} \frac{\ln\left(\frac{1}{x}\right)}{x} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{x\cdot\left(-\frac{1}{x^2}\right)}{1} = \lim_{x\to\infty} -\frac{1}{x} = 0 \implies \ln(L) = 0 \implies L = 1$$

The limit $\lim_{n\to\infty}(-1)^{n+1}$ does not exist because of the oscillation. So, $\lim_{n\to\infty}(-1)^{n+1}\left(\dfrac{1}{n}\right)^{1/n}$ does not exist as well. Therefore, by the nth Term Test for divergence, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{1}{n}\right)^{1/n}$ is divergent.

5. The Maclaurin series of f is given by $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$.

Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$ to look for the pattern.

$$f'(x) = \frac{1}{2\sqrt{e^x}}\cdot e^x = \frac{1}{2}\sqrt{e^x}, \quad f''(x) = \frac{1}{4\sqrt{e^x}}\cdot e^x = \frac{1}{4}\sqrt{e^x}, \quad f'''(x) = \frac{1}{8\sqrt{e^x}}\cdot e^x = \frac{1}{8}\sqrt{e^x}$$

$$f(0) = 1, \quad f'(0) = \frac{1}{2}, \quad f''(0) = \frac{1}{4}, \quad f'''(0) = \frac{1}{8}$$

Therefore, $f^{(k)}(0) = \left(\dfrac{1}{2}\right)^k$. Rewrite the summation formula.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^k \cdot x^k}{k!} = \boxed{\sum_{k=0}^{\infty} \frac{x^k}{2^k\cdot k!} = 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + ...}$$

Find the interval of convergence by applying the Ratio Test.

$$\lim_{k\to\infty}\left|\frac{x^{k+1}}{2^{k+1}\cdot(k+1)!}\cdot\frac{2^k\cdot k!}{x^k}\right| = \lim_{k\to\infty}\frac{x}{2k} = 0 < 1$$

$$\boxed{\text{The series is convergent on } \mathbb{R}.}$$

2024-2025 Makeup (31/01/2025) Solutions
(Last update: 17/08/2025 20:56)

1.

(a)

$$\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx = \int \frac{(1-\cos^2 x)\cdot \sin x}{\sqrt{\cos x}}\,dx$$

Let $u = \cos x$, then $du = -\sin x\,dx$.

$$\int \frac{(1-\cos^2 x)\cdot \sin x}{\sqrt{\cos x}}\,dx = \int -\frac{(1-u^2)}{\sqrt{u}}\,du = \int \left(u^{3/2} - \frac{1}{\sqrt{u}}\right) = \frac{2}{5}u^{5/2} - 2\sqrt{u} + c$$

$$= \boxed{\frac{2}{5}(\cos x)^{5/2} - 2\sqrt{\cos x} + c, \quad c \in \mathbb{R}}$$

(b) Use the method of partial fraction decomposition.

$$\int \frac{dx}{x^3 - 4x^2 + 3x} = \int \frac{dx}{x(x-3)(x-1)} = \int \left(\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-1}\right) dx$$

$$\begin{aligned} A(x-3)(x-1) + Bx(x-1) + Cx(x-3) &= 1 \\ x^2(A+B+C) + x(-4A-B-3C) + 3A &= 1 \end{aligned}$$

Equate the coefficients of like terms.

$$\left.\begin{aligned} x^2(A+B+C) &= 0 \\ x(-4A-B-3C) &= 0 \\ 3A &= 1 \end{aligned}\right\} \to A = \frac{1}{3}, \qquad \left.\begin{aligned} B + C &= -\frac{1}{3} \\ B + 3C &= -\frac{4}{3} \end{aligned}\right\} \to B = \frac{1}{6}, \quad C = -\frac{1}{2}$$

Rewrite the integral.

$$\int \left(\frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-1}\right) dx = \int \left(\frac{1}{3x} + \frac{1}{6(x-3)} - \frac{1}{2(x-1)}\right) dx$$

$$= \boxed{\frac{1}{3}\ln|x| + \frac{1}{6}\ln|x-3| - \frac{1}{2}\ln|x-1| + c, \quad c \in \mathbb{R}}$$

(c) The limit is in the indeterminate form 0/0. Apply L'Hôpital's rule to eliminate the indeterminate form.

$$\lim_{x \to \frac{\pi}{2}} \frac{\displaystyle\int_{\pi/2}^{x} \ln(\sin t)\,dt}{\sin x - 1} \overset{\text{L'H.}}{=} \lim_{x \to \frac{\pi}{2}} \frac{\displaystyle\frac{d}{dx}\int_{\pi/2}^{x} \ln(\sin t)\,dt}{\cos x}$$

By the Fundamental Theorem of Calculus, we may rewrite the limit as follows.

$$\lim_{x\to\frac{\pi}{2}} \frac{\dfrac{d}{dx}\displaystyle\int_{\pi/2}^{x} \ln(\sin t)\,dt}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{\ln(\sin x)}{\cos x} \overset{\text{L'H.}}{=} \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin x}\cdot\cos x}{-\sin x} = \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{-\sin^2 x} = -\frac{\cos\frac{\pi}{2}}{\sin^2\frac{\pi}{2}}$$

$$= \boxed{0}$$

(d) Take the limit as this is an improper integral.

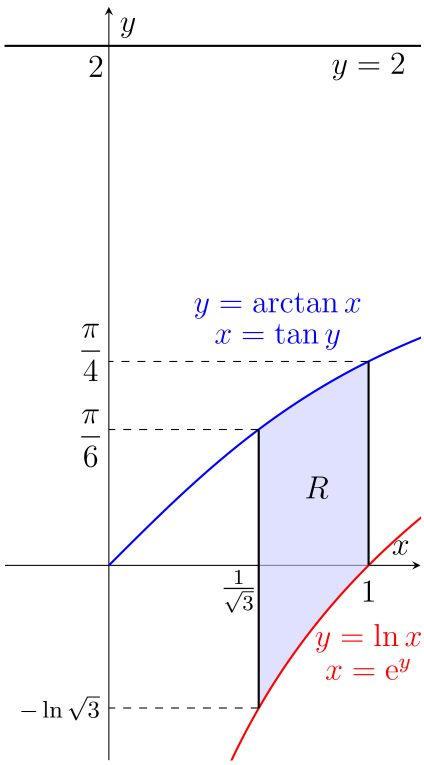
$$\int_0^2 \frac{dx}{(x-1)^{2/3}} = \lim_{R\to 1^-}\int_0^R \frac{dx}{(x-1)^{2/3}} + \lim_{P\to 1^+}\int_P^2 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{R\to 1^-} 3(x-1)^{1/3}\Big|_0^R + \lim_{P\to 1^+} 3(x-1)^{1/3}\Big|_P^2$$

$$= 3\lim_{R\to 1^-}\big((R-1)^{1/3}-(-1)\big) + 3\lim_{P\to 1^+}\big(1-(P-1)^{1/3}\big) = \boxed{6}$$

2.

(a)



$$A = \int_{1/\sqrt{3}}^{1} (\arctan x - \ln x)\;dx = \int_{1/\sqrt{3}}^{1} \arctan x\,dx - \int_{1/\sqrt{3}}^{1} \ln x\,dx \tag{38}$$

Calculate the first integral in (1) by integration by parts.

$$\left.\begin{array}{r}u=\arctan x \implies du=\dfrac{1}{x^2+1}\,dx \\ dv=dx \implies v=x\end{array}\right\} \to \int u\,dv = uv-\int v\,du$$

$$\int_{1/\sqrt{3}}^{1} \arctan x\,dx = x\arctan x\Big|_{1/\sqrt{3}}^{1} - \int_{1/\sqrt{3}}^{1}\frac{x}{x^2+1}\,dx = \left(x\arctan x-\frac{1}{2}\ln\left|x^2+1\right|\right)\Bigg|_{1/\sqrt{3}}^{1}$$

$$=\left(\frac{\pi}{4}-\frac{\ln 2}{2}\right)-\left(\frac{\pi\sqrt{3}}{18}-\frac{1}{2}\cdot\ln\left(\frac{4}{3}\right)\right)=\frac{\pi\left(9-2\sqrt{3}\right)}{36}+\frac{1}{2}\cdot\ln\frac{2}{3}$$

Calculate the second integral in (1) by integration by parts.

$$\left.\begin{array}{r}u=\ln x \implies du=\dfrac{1}{x}\,dx \\ dv=dx \implies v=x\end{array}\right\} \to \int u\,dv = uv-\int v\,du$$

$$\int_{1/\sqrt{3}}^{1} \ln x\,dx = x\ln x\Big|_{1/\sqrt{3}}^{1} - \int_{1/\sqrt{3}}^{1} dx = \left(x\ln x - x\right)\Bigg|_{1/\sqrt{3}}^{1} = (0-1)-\left(-\frac{\ln\sqrt{3}}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right)$$

$$=\frac{\sqrt{3}\ln\sqrt{3}+\sqrt{3}-3}{3}$$

The result is then

$$A=\boxed{\frac{\pi\left(9-2\sqrt{3}\right)}{36}+\frac{1}{2}\cdot\ln\frac{2}{3}-\frac{\sqrt{3}\ln\sqrt{3}+\sqrt{3}-3}{3}}$$

(b)

$$\boxed{\begin{array}{l}\displaystyle\int_{-\ln\sqrt{3}}^{0}\pi\left[\left(e^y\right)^2-\left(\frac{1}{\sqrt{3}}\right)^2\right]dy+\int_{0}^{\pi/6}\pi\left[(1)^2-\left(\frac{1}{\sqrt{3}}\right)^2\right]dy \\ \displaystyle+\int_{\pi/6}^{\pi/4}\pi\left[\left(1^2\right)-(\tan y)^2\right]\,dy\end{array}}$$

(c)

$$\boxed{\begin{array}{l}\displaystyle\int_{-\ln\sqrt{3}}^{0}2\pi\,(2-y)\left(e^y-\frac{1}{\sqrt{3}}\right)dy+\int_{0}^{\pi/6}2\pi\,(2-y)\left(1-\frac{1}{\sqrt{3}}\right)dy \\ \displaystyle+\int_{\pi/6}^{\pi/4}2\pi\,(2-y)\,(1-\tan y)\;dy\end{array}}$$

3.

(a)

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}} = \sum_{n=0}^{\infty} \frac{-3 \cdot (-3)^n}{8^n} = -3 \sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n$$

This is a geometric series where $r = -\frac{3}{8}$. $|r| = \frac{3}{8} < 1$. Therefore, the series $\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$ converges.

(b) Take $f(x) = \frac{1}{x(\ln x)^2}$. f is positive and decreasing for $x \geq 2$ because x and $(\ln x)^2$ are positive and increasing for $x \geq 2$. x is a polynomial which is defined everywhere and $(\ln x)^2$ is continuous for $x \geq 2$. Since we took into account every criterion, we may apply the Integral Test. Handle the improper integrals by taking the limit.

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{R \to \infty} \int_2^R \frac{dx}{x(\ln x)^2} = \lim_{R \to \infty} \left[-\frac{1}{\ln x}\right]_2^R = \lim_{R \to \infty} \left[-\frac{1}{\ln R} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$$

Since the integral converges, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ also converges.

4. The Taylor series of f at $c = 1$ is as follows.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

Find $f(1)$, $f'(1)$, $f''(1)$, $f'''(1)$ $f^{(4)}(1)$ to look for the pattern.

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f'''(x) = \frac{2}{x^3} \quad f^{(4)}(x) = -\frac{6}{x^4}$$

$$f(1) = 0, \quad f'(1) = 1, \quad f''(1) = -1, \quad f'''(1) = 2, \quad f^{(4)}(1) = -6$$

This is an alternating sequence where the coefficient of each term is the factorial of the subsequent number starting from 0 except for $k = 0$, that is, the first term of the series. At $k = 0$, the first term is 0. So,

$$f^k(1) = \begin{cases} (-1)^{k-1} \cdot (k-1)!, & \text{if } k > 0 \\ 0, & \text{if } k = 0 \end{cases}$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} x^k = 0 + \sum_{k=1}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot (k-1)!}{k \cdot (k-1)!}(x-1)^k$$

$$= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot (x-1)^k}{k} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + ...}$$

Now, determine the interval of convergence. Apply the Ratio Test.

$$\lim_{k\to\infty}\left|\frac{(-1)^k(x-1)^{k+1}}{k+1}\cdot\frac{k}{(-1)^{k-1}(x-1)^k}\right| = \lim_{k\to\infty}\left|\frac{(x-1)\cdot k}{(k+1)\cdot(-1)}\right| = |x-1|\lim_{k\to\infty}\left|\frac{k}{k+1}\right|$$

$$= |x-1|$$

$$|x-1|<1 \implies -1<x-1<1 \implies 0<x<2 \quad \text{(convergent)}$$

Investigate the convergence at the endpoints.

$$x=0 \to \sum_{k=1}^{\infty}\frac{(-1)^{k+1}\cdot(-1)^k}{k} = \sum_{k=1}^{\infty}\frac{(-1)^{2k}\cdot(-1)}{k} = -\sum_{k=1}^{\infty}\frac{1}{k}$$

This is a p-series with $p=1$, for which the series diverges by the p-series Test. Try $x=2$.

$$x=2 \to \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$$

This is an alternating series. The non-alternating part, which is $\frac{1}{k}$, is nonincreasing for $k\geq 1$ and it is positive. The limit at infinity is 0. By Leibniz's Alternating Series Test, the series converges.

The convergence set for the power series is $\boxed{(0,2]}$.